Parry numbers and non-Parry numbers

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$\mathbb{R}^n$ ambiant nth-dim Euclidean space.

The set of uniformly discrete sets of the same constant is metric and compact (Thm : Muraz, VG 2005 - as a generalization of the Compacity Thm of K. Mahler).

Definition (NEW)

A UD-set is a point set which is uniformly discrete on each compact.

Same for Delone sets. The “small Delone constant" depends upon the compact.

In Mathematics :

Freeman Dyson : Riemann Hypothesis

"zeros on the critical line are a 1d quasicrystal".

Freeman Dyson : *Random Matrices, Neutron Capture Levels, Quasicrystals and Zeta-functions zeros*.

Talk given at Mathematical Science Research Institute, Workshop on Random Matrix Theory, berkeley, California, September 23 2002.

http ://www.math.ucdavis.edu/ oyounggo/books/dyson.pdf

Other speculative articles, more recent

"Indian Mathematician Attempts To Solve Riemann Hypothesis Using Quasicrystals ¿ And He Wants You In Too !" By Debjyoti Bardhan on October 12th, 2011.

techie-buzz(http ://techie-buzz.com/science/indian-riemann-hypothesis-quasicrystals.html)

P. de la Harpe (mathematical comments on QCs and the Nobel prize of Dan Schechtman 2011)

Beta-integers :

for $\beta > 1$ an algebraic number :

$$\mathbb{Z}_\beta$$

$\beta = (1 + \sqrt{5})/2$: Penrose tilings, 2d, 3d.

β : Parry number : $\mathbb{Z}_\beta$ Meyer set, ie uniformly discrete and $\mathbb{Z}_\beta - \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F$, F finite.

β non-Parry number : UD-set in general, not uniformly discrete, with $\beta\mathbb{Z}_\beta \subset \mathbb{Z}_\beta$.
Meyer, Lagarias ; arithmetics -> self-similar, self-affine (Pisot, Salem, ...).

β-integers are polynomials in β with integer coefficients : as such belong to the number field $\mathbb{Q}(\beta)$.

use the embedding in the ring of adèles (topological ring) :

$$\text{lattice in} \qquad \mathbb{K} := \mathbb{Q}(\beta) \to \mathbb{A}_{\mathbb{K}}$$

to obtain points on the “lattice" by lifting. “Cut-and-project scheme" machinery.

“arithmetical translations" + completion : topological dynamics : operators on the fundamental domain

$$\mathbb{A}_{\overline{\mathbb{K}}}/\overline{\mathbb{K}}.$$

In Physics/Crystallography :

New mathematical definition for UD-sets :

coherent with experiments : sample under incident radiation : always FINITE number of atom sites + X-ray Diffraction theory (Laué).

NO NEED to have point sets constructed in mathematics having the uniformly discreteness property at "infinity" with the same constant.

Contents

Algebraicity and digits :

$$\beta = \text{base of numeration (Rényi)}$$

Context : relations between $\beta > 1$ an algebraic number, the coefficients of its minimal polynomial, the digits t_i in the Rényi β-expansion of

$$\beta = \beta d_\beta(1) = \beta \times 0.t_1 t_2 t_3 \ldots \qquad ?$$

Dynamical system of numeration : $([0,1], T_\beta)$, with $T_\beta : x \to \{\beta x\}$ the beta-transformation.

$x > 0$ real, represented as (Laurent series)

$$x = \sum_{i=m}^{+\infty} a_i \beta^{-i}, \qquad a_i \in \mathbb{Z}.$$

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Rényi, Parry (1960) : Unique representation when the sequence (a_i) satisfies the conditions of Parry, with $a_i \in \{0, 1, \ldots, \lceil \beta - 1 \rceil\}$, alphabet of the β-shift.

$$x = \sum_{i=m}^{0} a_i \beta^{-i}, \qquad m \leq 0, \qquad \mathrm{beta-integer}$$

Conditions of Parry : sequence (t_i) of digits, from the orbit of 1 under the iterates of T_β : $t_1 = \lfloor \beta \rfloor, t_2 = \lfloor \beta\{\beta\} \rfloor = \lfloor \beta T_\beta(1) \rfloor, t_3 = \lfloor \beta\{\beta\{\beta\}\} \rfloor = \lfloor \beta T_\beta^2(1) \rfloor, \ldots$.

Analytic functions constructed on it : Dynamical zeta function $\zeta_\beta(z)$; Parry Upper function :

$$f_\beta(z) := -1 + \sum_{i=1}^{+\infty} t_i z^i .$$

Relations (only for Parry numbers) : minimal polynomial $P_\beta(X)$, its Parry polynomial $P_{\beta,P}(X) \in P_\beta(X)\mathbb{Z}[X]$ (with $P^*_{\beta,P}$ denoting its reciprocal polynomial), as

$$f_\beta(z) = -\frac{1}{\zeta_\beta(z)} = -\frac{P^*_{\beta,P}(z)}{(1-z^{p+1})} \qquad \text{nonsimple } \beta$$

where $p+1$ is the period length, and

$$f_\beta(z) = -\frac{1-z^m}{\zeta_\beta(z)} = -P^*_{\beta,P}(z) \qquad \text{simple } \beta$$

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non-Parry ? Rényi-Parry germ of curve.

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A 2-variable convergent formal series (germ)

Theorem

Let $d := \deg \beta$. *There exists an unique* $G_\beta(U,Z) \in \mathbb{C}[[U]][Z]$ *with* $\deg_Z(G_\beta(U,Z)) \leq d-1$ *such that*

$$G_\beta(P_\beta^*(z), z - 1/\beta) = f_\beta(z).$$

Here $Z := z - 1/\beta$, the origin of $\mathbb{C}^2$ is $(0, 1/\beta)$ (variables : U and z).

-> $f_\beta(z)$ as a 2-variable Taylor series parametrized by $(P_\beta^*(z), z - 1/\beta)$ in a neighbourhood of $1/\beta$.

$$G_\beta(U,Z) = a_{d-1}(U)Z^{d-1} + a_{d-2}(U)Z^{d-2} + \ldots + a_1(U)Z + a_0(U)$$

with $a_j(U) \in \mathbb{C}[[U]], j = 0, 1, \ldots, d-1$.

We can always assume it is distinguished (i.e. the orders of $a_j(U)$ are all ≥ 1), by the

Weierstrass theorem (Weierstrass polynomial) :

$$G_\beta(U,Z) = \text{unit} \times \text{distinguished polynomial}$$

method of algebraic geometry
curves (dim 1)
desingularisation
Theory of Puiseux

[VG, Integers (2011) ; (2012)].

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what is a plane algebraic curve ?

$$\mathcal{C}_\beta := \{(U,Z) \in \mathbb{C}^2 \mid G_\beta(U,Z) = \sum_{m,n\geq 0} A_{m,n} Z^m U^n = 0\}$$

with coefficients $A_{m,n}$ in a number field $\mathbb{K}$, with a ramified covering

$$\pi_\beta : \mathcal{C}_\beta \to \mathbb{C}$$

of $\mathbb{C}$ (i.e. the U-plane).

R-P : $\sum_{m,n\geq 0} A_{m,n} Z^m U^n = \sum_{m=0}^{\delta}(\sum_{n\geq 0} A_{m,n} U^n) Z^m$.

A point $b \in \mathbb{C}$ is called

- regular if the fiber $\pi_\beta^{-1}(b)$ has cardinality δ, and
- critical if it is not the case.

The number of critical points is finite (Pham, Lefshetz).

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Puiseux theorem, using the Newton polygon (Puiseux, 1851 ; Casas-Alvero, Duval, Walker, Pham, ...).

If y is a Puiseux series, write $g_y = \prod_{i=1}^{\nu(y)}(X - y_i(Y))$, the $y_i, i = 1, \ldots, \nu(y)$ being the conjugates of y.

Theorem

For any $g(X, Y) \in \mathbb{C}[[X, Y]]$,

(i) *there are Puiseux series $y_1, y_2, \ldots, y_m, m \geq 0$, so that g decomposes in the form*

$$g = u\, Y^r\, g_{y_1}\, g_{y_2} \ldots g_{y_m}$$

where $r \in \mathbb{N}$, and u is an invertible series in $\mathbb{C}[[X, Y]]$,

(ii) *the height of the Newton polygon of g is*

$$h(N(g)) = \nu(y_1) + \nu(y_2) + \ldots + \nu(y_m)$$

and the X-roots of g are the conjugates of the $y_j(Y), j = 1, \ldots, m$.

non-parry β : only 3 cases. One unique conjugacy class of Puiseux series.

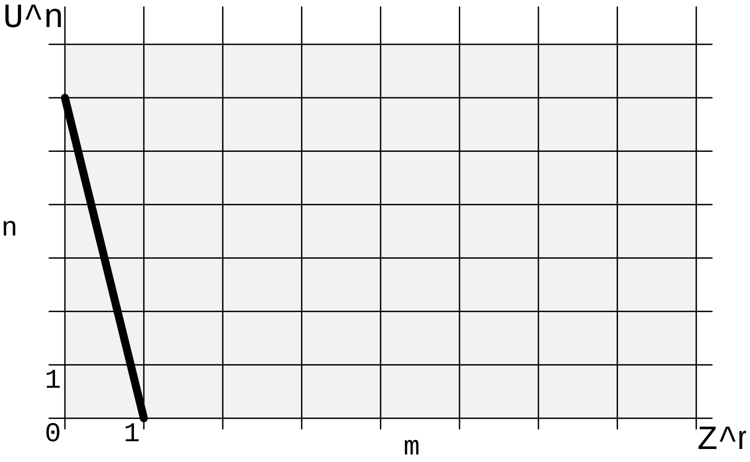

FIGURE: Rényi numeration : Newton polygon - type 1.

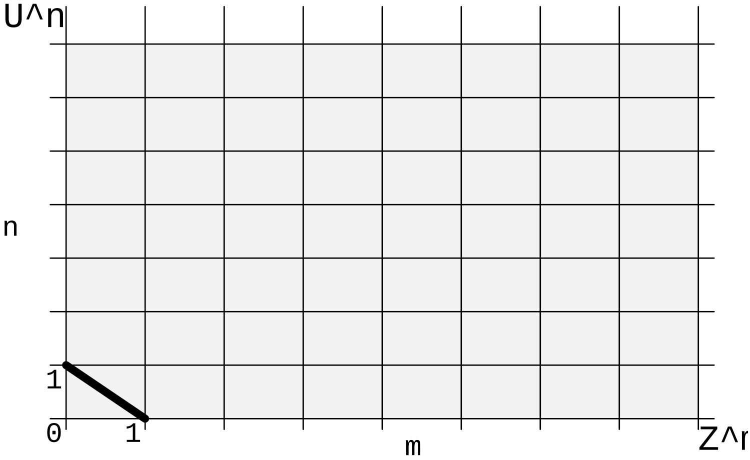

FIGURE: Rényi numeration : Newton polygon - type 2.

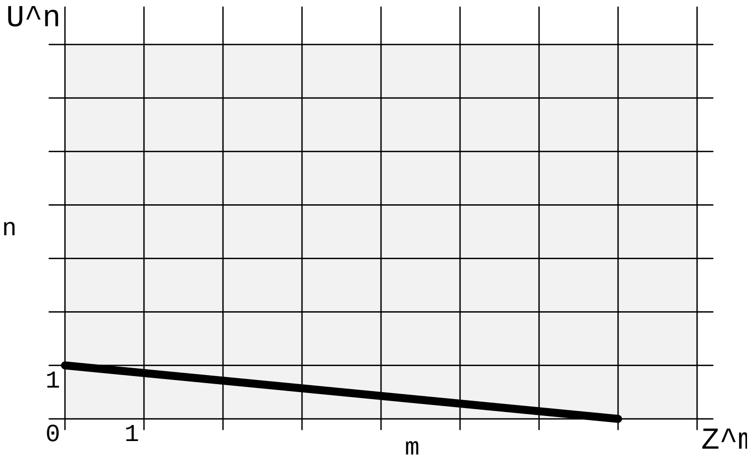

FIGURE: Rényi numeration : Newton polygon - type 3.

Beta-conjugate of β

$$\omega = \frac{1}{\beta} + \sum_{i \geq 0} h_i ((P_\beta^*(\omega))^{1/n})^i$$

with Galois action (ξ n-th root of unity) :

$$\omega' = \frac{1}{\beta} + \sum_{i \geq 0} h_i \xi^i ((P_\beta^*(\omega'))^{1/n})^i.$$

Tile of $\mathbb{Z}_\beta$:= interval between two successive beta-integers.

length : fully formulable as a function of β, the derivatives of the minimal polynomial of β at β, the coefficients appearing in the decomposition of the germ G_β (Newton polygon).

Infinite number of tiles.

Classification of non-Parry numbers : based on the degree of the Weierstrass polynomial arising in the decomposition of the germ.