A modified dual-tiling approach to the Anderson-Putnam method

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Joint work with

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The Hull of a Tiling

Let $\mathcal{T}$ be a tiling of $\mathbb{R}^d$, of finite local complexity.

We introduce a topology on a set of tilings using a metric:

Two tilings have distance $< \epsilon$, if they agree in a ball of radius $1/\epsilon$ around the origin, up to a translation $< \epsilon$.

The hull Ω is then the closure of $\{\mathcal{T} - x | x \in \mathbb{R}^d\}$.

Ω is a compact metric space, on which $\mathbb{R}^d$ acts by translation.

If $\mathcal{T}$ is repetitive, every orbit is dense in Ω.

Ω then consists of the LI class of $\mathcal{T}$.

Approximating the Hull by Finite Cell Complexes

We define a sequence of cellular (CW-)spaces Ω_n approximating the hull Ω.

We first label the tiles according to their first corona – we collar them.

The d-cells of Ω_0 are then the interiors of the tiles; two tile boundaries are identified if they are shared somewhere in the tiling.

Points in Ω_0 represents a cylinder set of tilings, whose origin is in a given tile at a given position. For Ω_n we do the same with n-th order supertiles.

There are natural, continuous cellular mappings $h : \Omega_n \to \Omega_{n-1}$, and induced homomorphisms $h_* : H^*(\Omega_{n-1}) \to H^*(\Omega_n)$.

Ω then is the inverse limit $\varprojlim \Omega_n$, consisting of all sequences $\{x_k\}_{k=0}^{\infty}$, with $x_k \in \Omega_k$ and $h(x_k) = x_{k-1}$.

Cohomology of Substitution Tilings

In fact, all complexes Ω_n can be identified, and we have a single map acting on a single complex Ω_0.

The Čech cohomology of Ω then is the direct limit $H^*(\Omega) \cong \varinjlim H^*(\Omega_0)$ under the induced map h_*.

This construction was introduced by Anderson and Putnam, Ergod. Th. & Dynam. Sys. 18, 509 (1998).

If the substitution does force the border, collaring is not necessary, and the construction simplifies.

Simplifying the Anderson-Putnam Construction

If the substitution forces the border, there is not much to be simplified.

Otherwise, we need collored tiles, and their number grows rapidly with dimension, especially for simplicial tilings.

Barge-Diamond (1d) and Barge-Diamond-Hunton-Sadun (any d) expand boundary cells to full dimension, and thereby save collaring.

G-Maloney introduce one-sided collaring in 1d (arXiv 1112.1475).

Correctly interpreted, this is the germ for a dual tiling approach.

Dual Tiling Approach

For a tiling $\mathcal{T}$, define a combinatorial dual tiling $\mathcal{T}^*$, by setting for the dual of a cell c the patch

$$c^* = \{T \in \mathcal{T} | c \subset T\}$$

Identifying translation equivalent cells, we obtain an analogue of the Anderson-Putnam complex.

To define a substitution for the dual tiling, we have to assign each dual tile to a unique dual supertile. In other words, we have to assing each (original) vertex to a unique supervertex.

This must be done in such a way, that the so defined dual supertiling is indeed dual to the supertile tiling.

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Dual Tiling Substitution Forces the Border

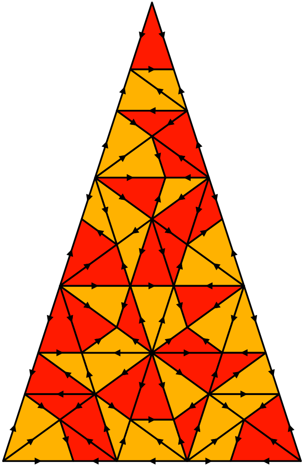
The collection of dual tiles belonging to a dual supertile may have a different geometry than the supertile. This can be fixed by applying a homotopy in the same way as Barge-Diamond do, without modifying the cell complex.

The key feature is: dual tiling substitution forces the border!

Reason: Patch belonging to a dual tile is strictly larger than the dual tile, and so has information on the surroundings.

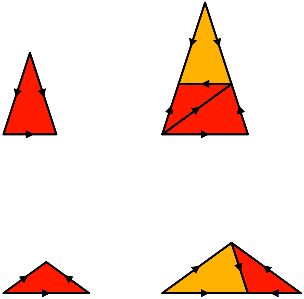
Note: Dualising twice yields the the original tiling, but with full collaring. Dualisation adds, in a sense, the square root of a full collaring.

Tübingen Triangle Tiling



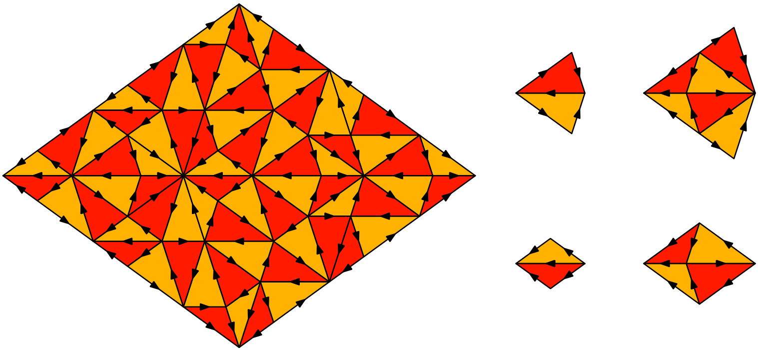
Klitzing, Schlottmann, Baake,

Int. J. Mod. Phys. B7 (1993) 1455–1473

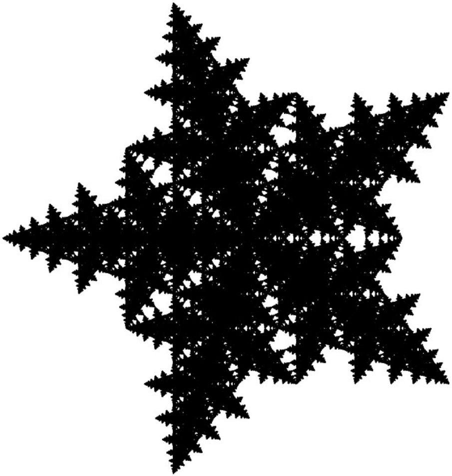


Danzer & Nischke $\mathrm{infl}_{2,-}$

Danzer & Nischke, Discrete Comput. Geom. **96**, 221-236

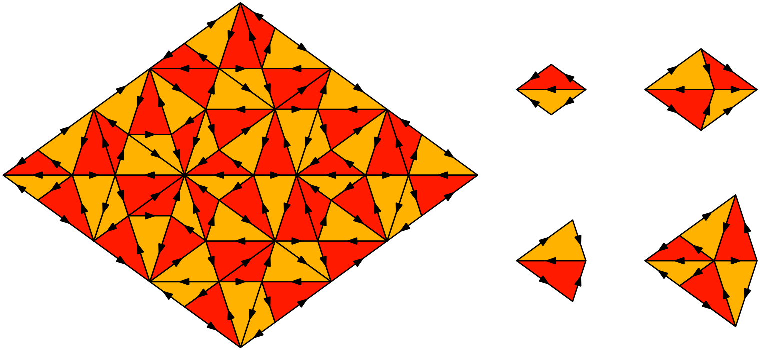


Window of $\mathrm{infl}_{2,-}$



Danzer & Nischke $\mathrm{infl}_{2,+}$

Danzer & K.P. Nischke, Discrete Comput. Geom. **96**, 221-236



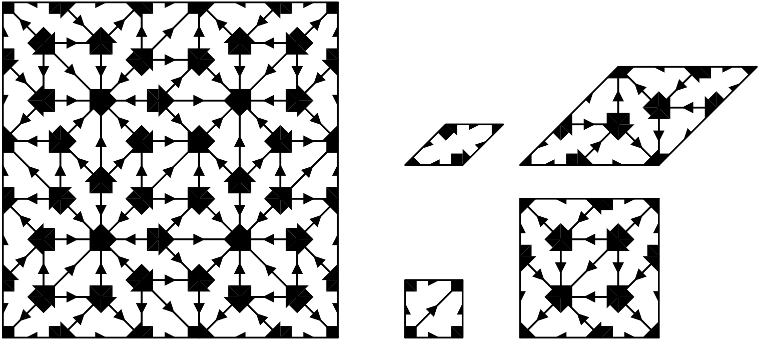
Cohomology of Triangle Tilings

Tübingen	0	1	2
$\mathrm{rk}\, C^k(\Gamma)$	40	140	130
$H^k(\Omega)$	$\mathbb{Z}$	$\mathbb{Z}^5$	$\mathbb{Z}^{24} \oplus \mathbb{Z}_2^2$

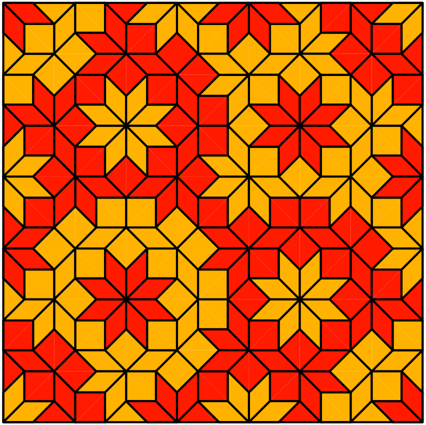
$\mathrm{infl}_{2,-}$	0	1	2
$\mathrm{rk}\, C^k(\Gamma)$	20	65	71
$H^k(\Omega)$	$\mathbb{Z}$	$\mathbb{Z}^4$	$\mathbb{Z}^{14}$

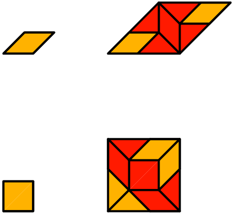
$\mathrm{infl}_{2,+}$	0	1	2
$\mathrm{rk}\, C^k(\Gamma)$	20	65	75
$H^k(\Omega)$	$\mathbb{Z}$	$\mathbb{Z}^5$	$\mathbb{Z}^{34}$

Octagonal Tiling with Ammann Decoration



Octagonal Tiling with Color Symmetry





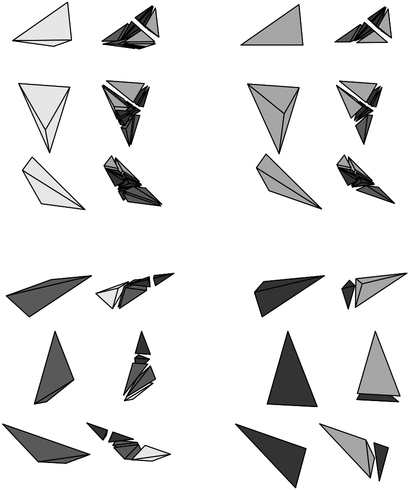
Cohomology of Octagonal Tilings

plain	0	1	2
$\mathrm{rk}\, C^k(\Gamma)$	24	60	49
$H^k(\Omega)$	$\mathbb{Z}$	$\mathbb{Z}^5$	$\mathbb{Z}^9$

decorated	0	1	2
$\mathrm{rk}\, C^k(\Gamma)$	32	144	152
$H^k(\Omega)$	$\mathbb{Z}$	$\mathbb{Z}^8$	$\mathbb{Z}^{23}$

colored	0	1	2
$\mathrm{rk}\, C^k(\Gamma)$	48	120	98
$H^k(\Omega)$	$\mathbb{Z}$	$\mathbb{Z}^5$	$\mathbb{Z}^{14} \oplus \mathbb{Z}_2$

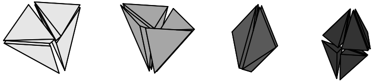
Danzer Tiling



L. Danzer, Discr. Math. **76** (1989) 1–7

Tetrahedra tiling with τ scaling

Can combine tetrahedra to octahedra:



$\rightarrow$ simple geometric matching rules

	0	1	2	3
$\mathrm{rk}\, C^k(\Gamma)$	480	1320	1320	480
$H^k(\Omega)$	$\mathbb{Z}$	$\mathbb{Z}^7$	$\mathbb{Z}^{16}$	$\mathbb{Z}^{20}$

Agrees with G-Hunton-Kellendonk

(arXiv 1202.2240)

Acknowledgement

A computer implementation of the substitution for the Danzer tiling provided by Uwe Grimm was of great help for the computation of the cohomology of the Danzer tiling.