When Periodicity Enforces Aperiodicity

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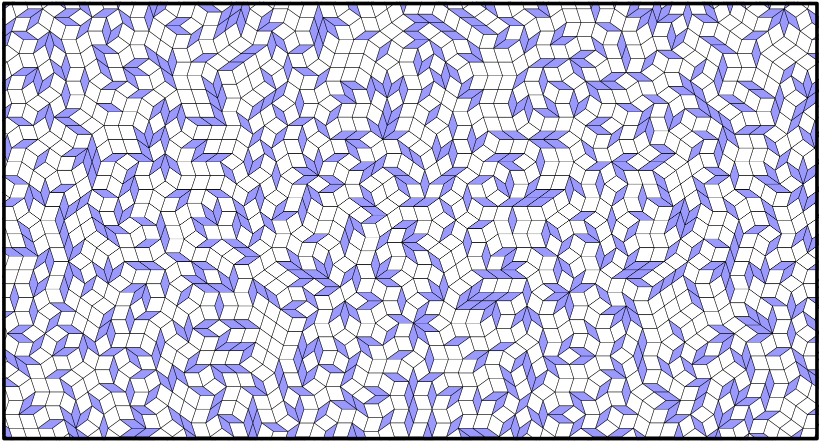
Marseille, january 17th, 2013

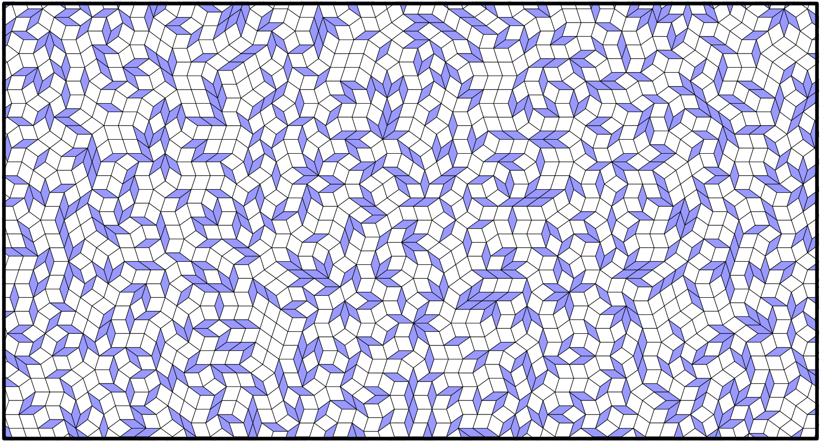
1. Settings
2. Main result
3. Examples

Planar rhombus tilings



n pairwise non-colinear vectors of $\mathbb{R}^2 \rightsquigarrow$ tilings of $\mathbb{R}^2$ by $\binom{n}{2}$ rhombi.

Planar rhombus tilings



Lift: homeomorphism which maps tiles on 2-faces of unit n-cubes.

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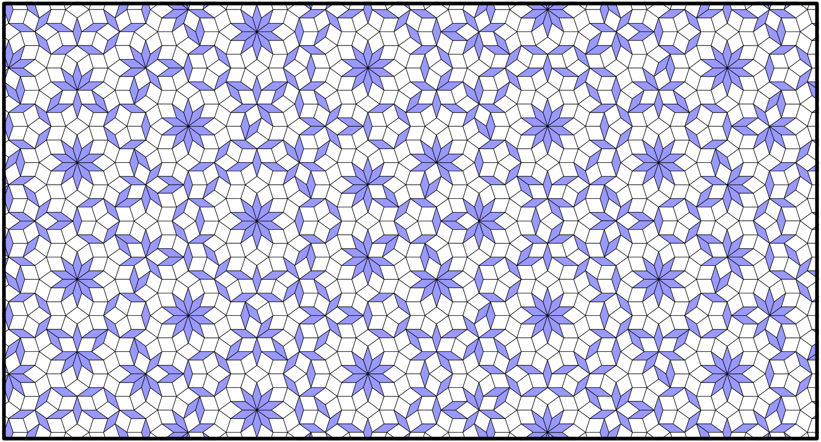


Planar: lift in $E + [0, t]^n$, where E is the *slope* and t the *thickness*.

Local rules

Definition

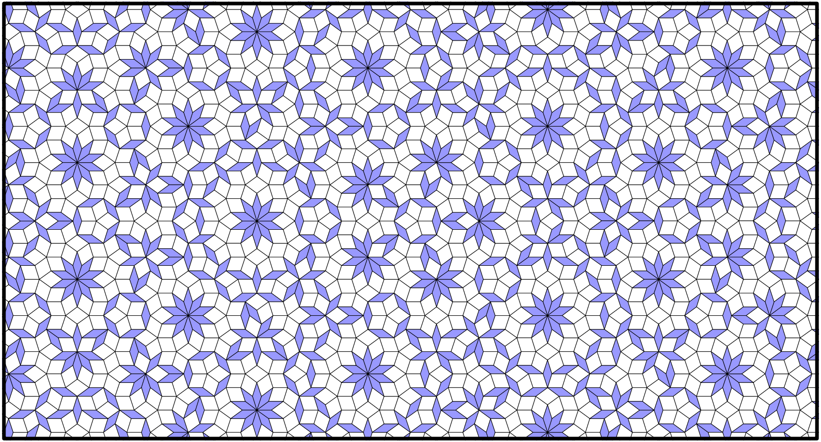
A slope E has *local rules* (LR) if there is a finite set of *patches* s. t. any rhombus tiling without any such patch is planar with slope E.

LR are said to be

- *strong* if the tilings satisfying them have thickness 1;
- *natural* if the thickness 1 tilings satisfy them;
- *weak* otherwise (the thickness is just bounded).

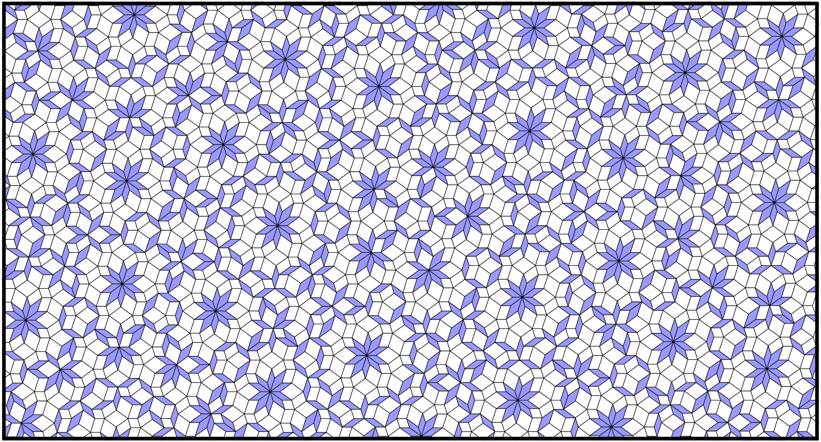
Mathieu's talk focused on weak LR. We here focus on natural LR.

Shadows and subperiods



Shadow: projection on a space generated by three basis vectors.

Shadows and subperiods



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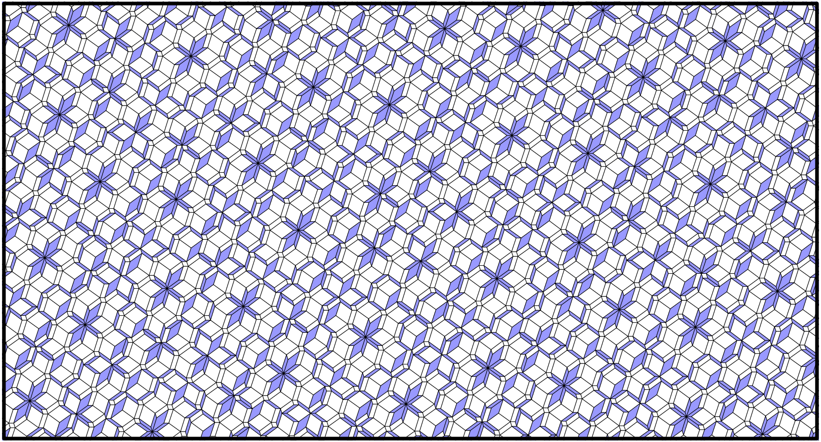

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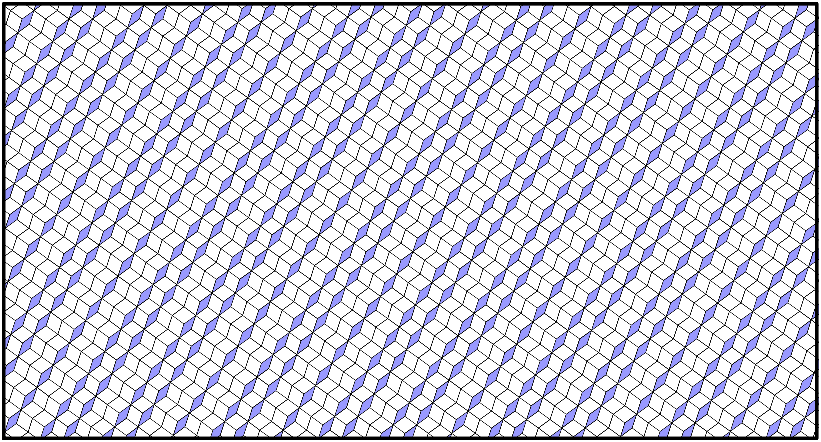

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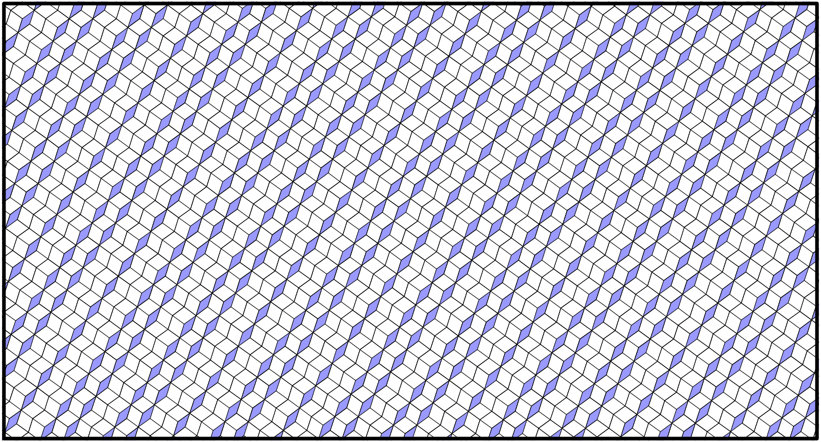

Subperiod: shadow period (tiling); shadow rational vector (slope).

A Characterization

Theorem

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Slopes with natural LR must be algebraic (Le'95). Here, we refine:

Corollary

A slope with natural LR is generated by vectors defined over a number field of degree at most $\lfloor \frac{n}{2} \rfloor$. *Degree* $\lfloor \frac{\phi(n)}{2} \rfloor$ *is reached.*

Necessity (sketch)

Definition

The set of singular points of order k of E is $\mathrm{Sing}_k(E) := E + \mathbb{Z}_k^n$.

Lemma

$Sing_k(E)$ cuts up the window *into convex connected components corresponding to "size k" patches of slope E thickness* 1 *tilings.*

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Lemma

Subperiods characterize either finitely many slopes, or a continuum.

Lemma

Subperiod $\Leftrightarrow$ intersection of boundaries of connected component.

Sufficiency (sketch)

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A slope satisfies the *P-condition* if it contains three non-collinear vectors which project onto subperiods in three irrational shadows.

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P-condition $\Leftrightarrow$ *planarity of the tilings with the same subperiods.*

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Lemma

Subperiods characterize finitely many slopes $\Rightarrow$ *P-condition holds.*

Grassmann-Plücker coordinates

Definition

The plane $\mathbb{R}\vec{u} + \mathbb{R}\vec{v}$ has GP-coordinates $(G_{ij})_{i<j} = (u_i v_j - u_j v_i)_{i<j}$.

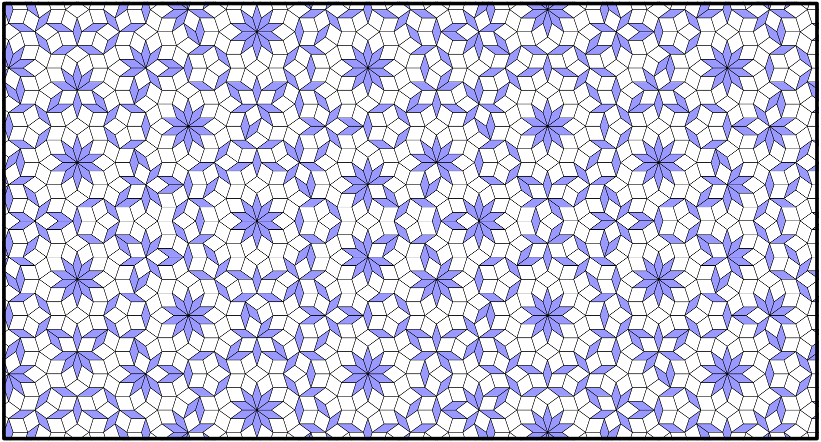
Proposition (Grassmann-Plücker)

GP-coordinates satisfy all the relations $G_{ij}G_{kl} = G_{ik}G_{jl} - G_{il}G_{jk}$.

Proposition

Whenever a planar tiling admits $p\vec{e}_i + q\vec{e}_j + r\vec{e}_k$ *as a subperiod, the GP-coordinates of its slope satisfy* $pG_{jk} - qG_{ik} + rG_{ij} = 0$.

Generalized Penrose tilings



The slope has GP-coordinates $(\varphi, 1, -1, -\varphi, \varphi, 1, -1, \varphi, 1, \varphi)$.

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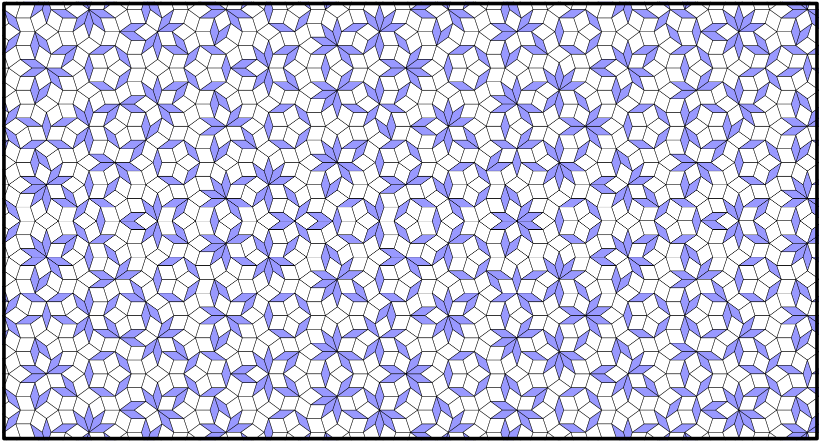

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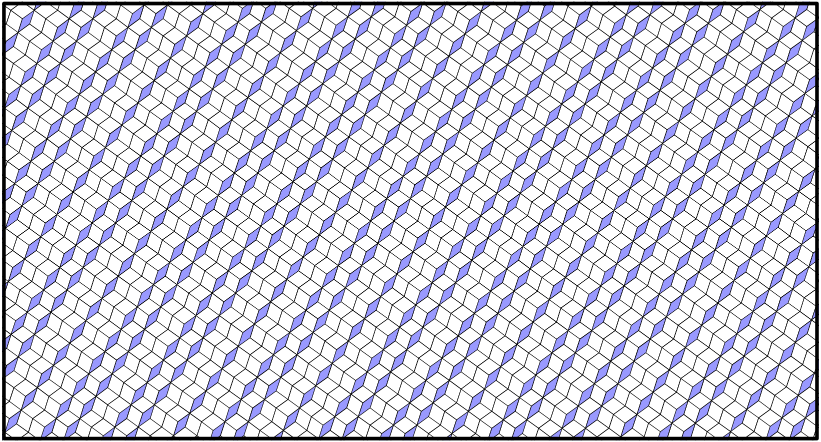

Subperiods yield $\begin{cases} G_{13} = G_{41} = G_{24} = G_{52} = G_{35} = 1 \\ G_{12} = G_{51} = G_{45} = G_{34} = G_{23} =: x \end{cases}$

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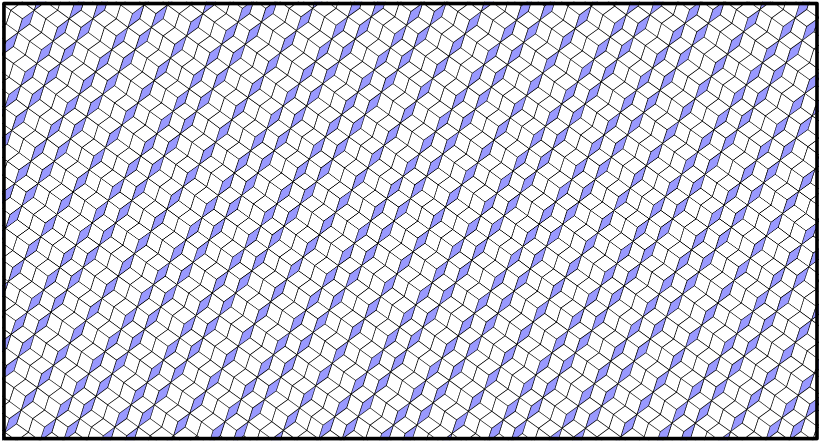

Plugged into the five GP-relations, this yields $x^2 = x + 1$.

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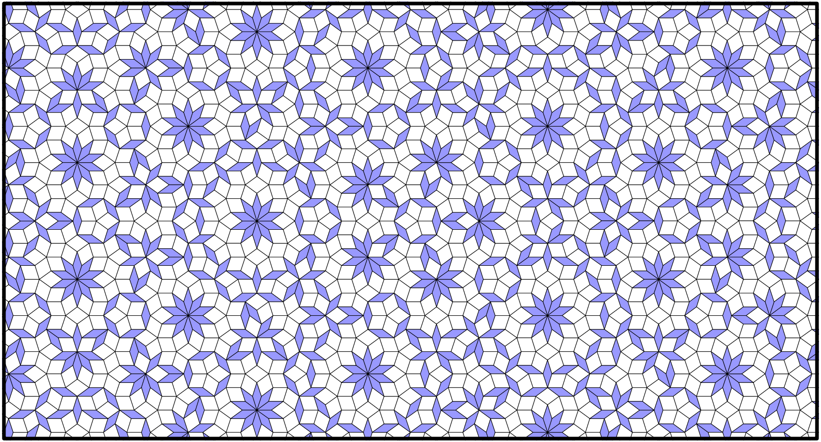

Subperiods characterize finitely many slopes: the theorem applies!

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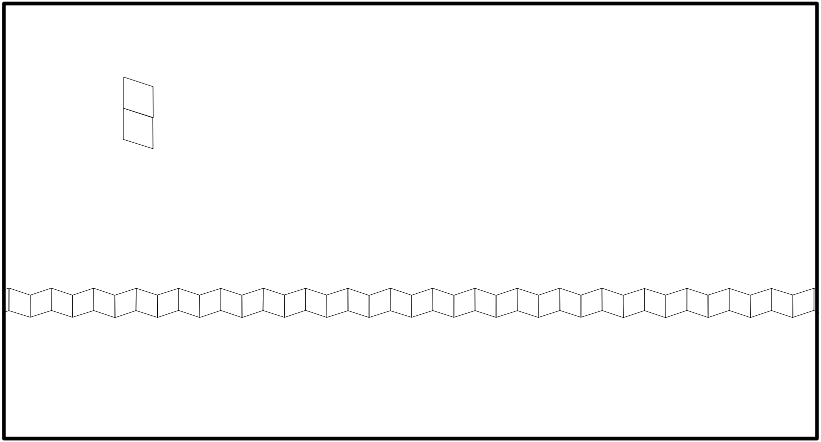

Subperiods are easily enforced in each shadow by forbidden patches.

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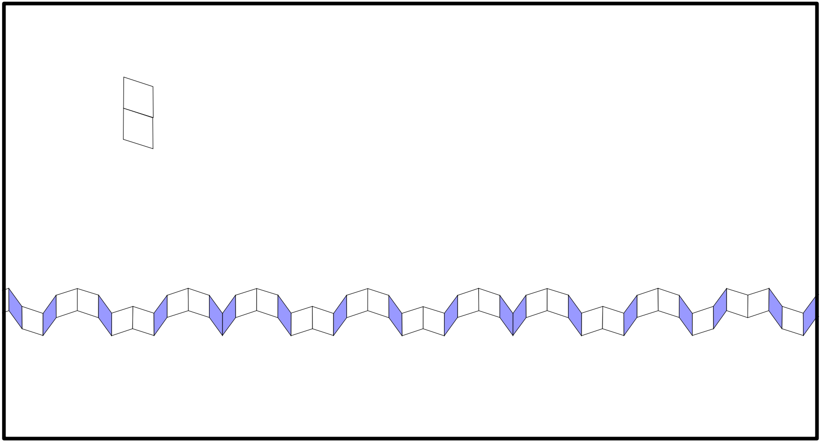

This still holds in the tilings, at least in those of thickness 1.

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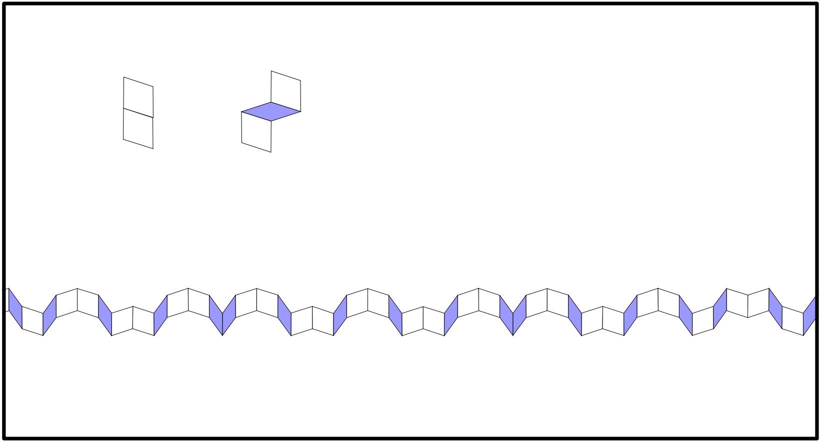

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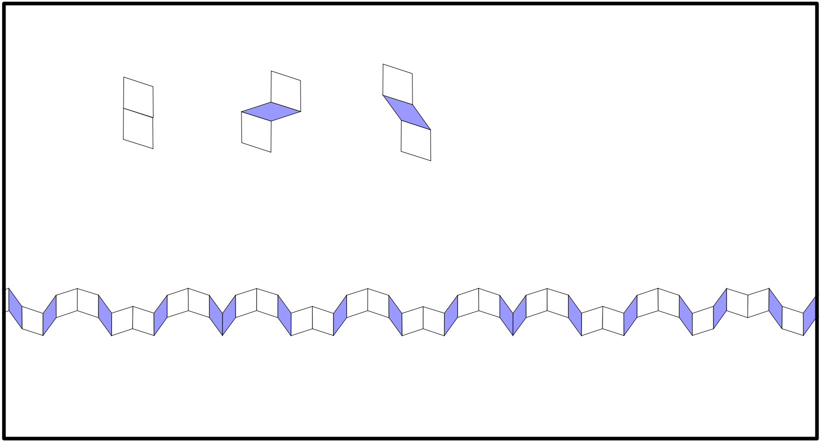

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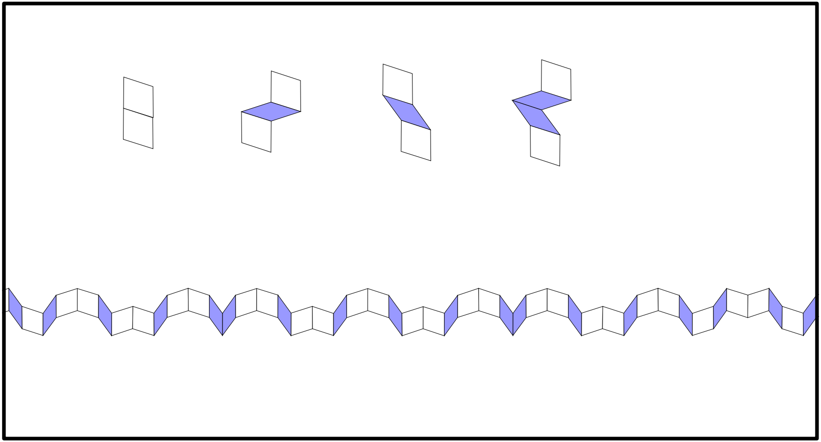

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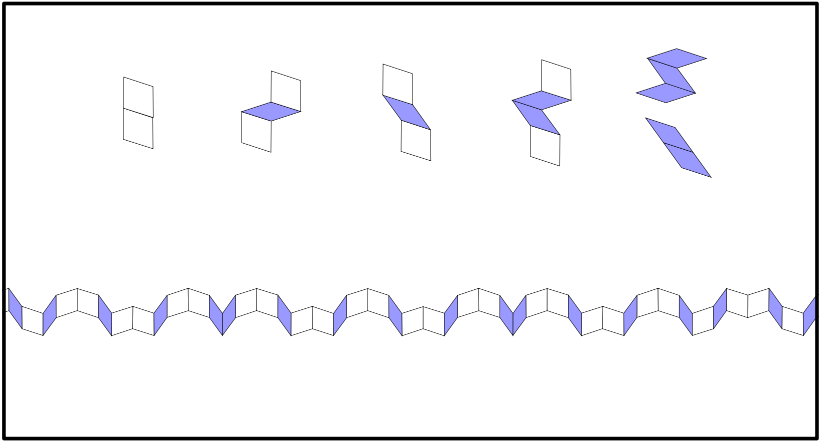

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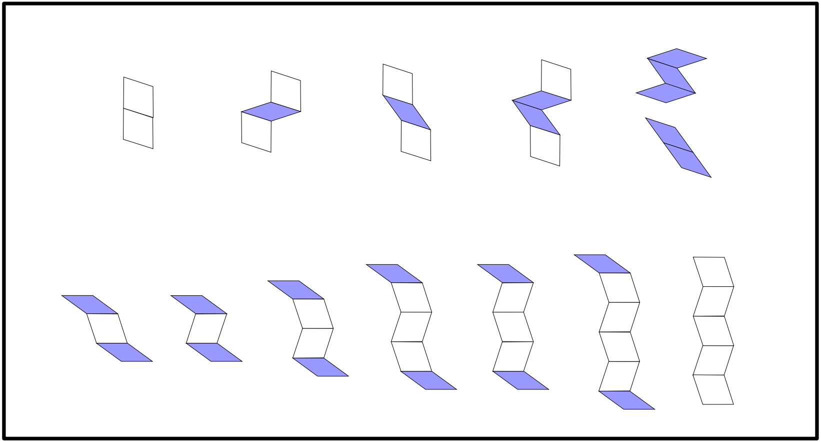
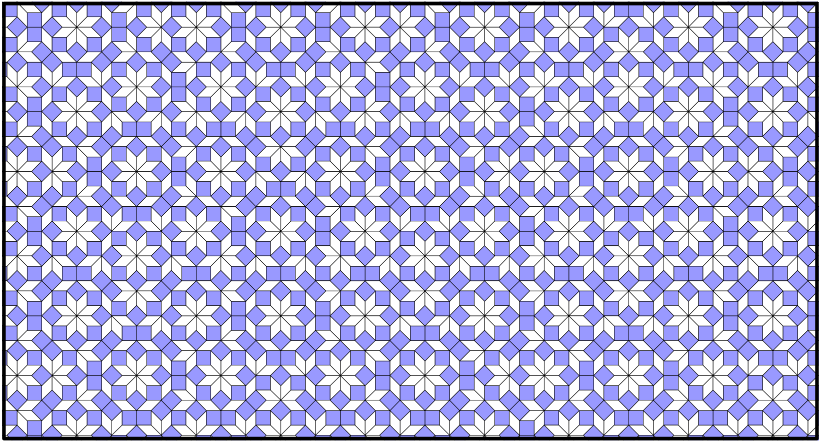


Considering all the shadows yields (simple) natural LR for the tilings.

Ammann-Beenker tilings



The slope has GP-coordinates $(1, \sqrt{2}, 1, 1, \sqrt{2}, 1)$.

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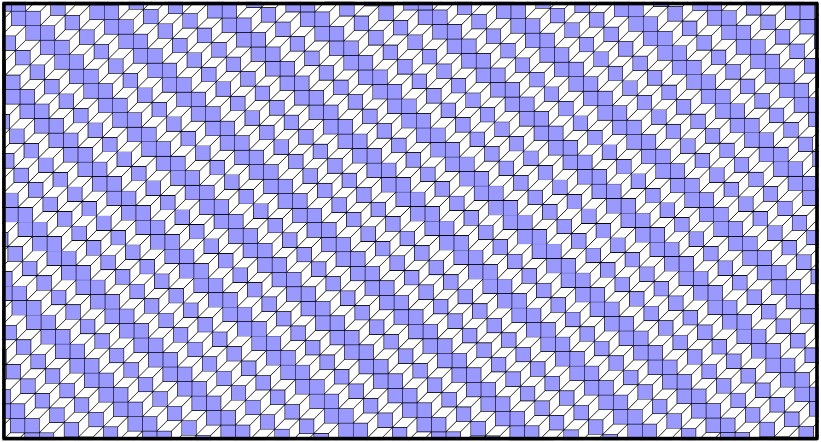

Subperiods yield $G_{12} = G_{14} = G_{23} = G_{34}$; GP-relation $G_{13}G_{24} = 2$.

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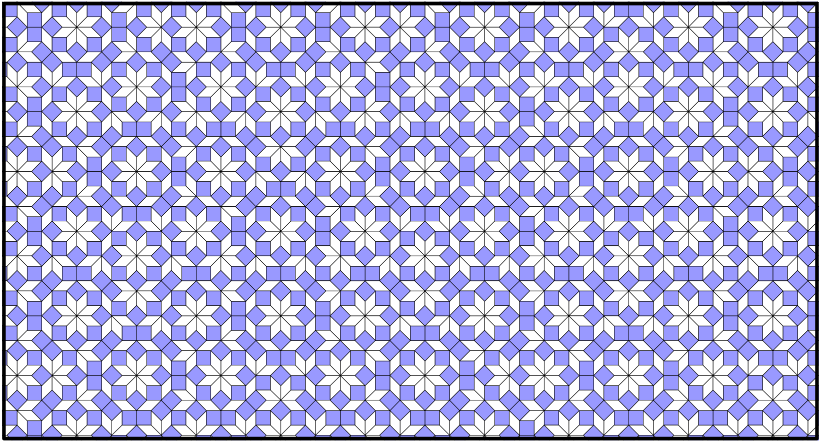

Subperiods thus characterize all the slopes $(1, t, 1, 1, 2/t, 1)$, $t \in \mathbb{R}$.

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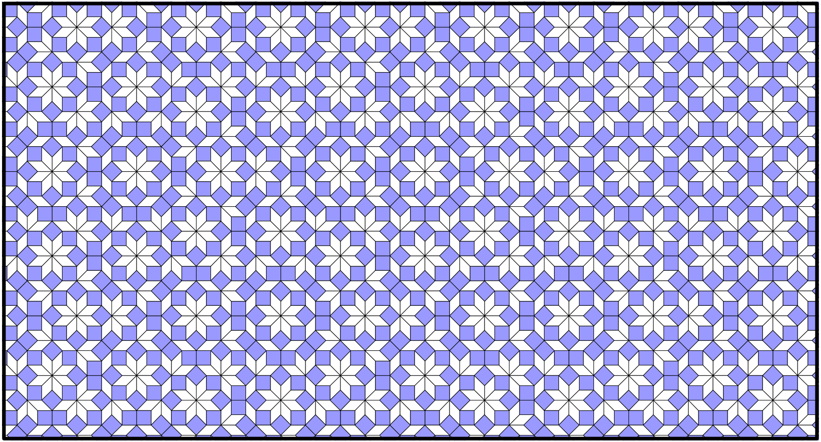

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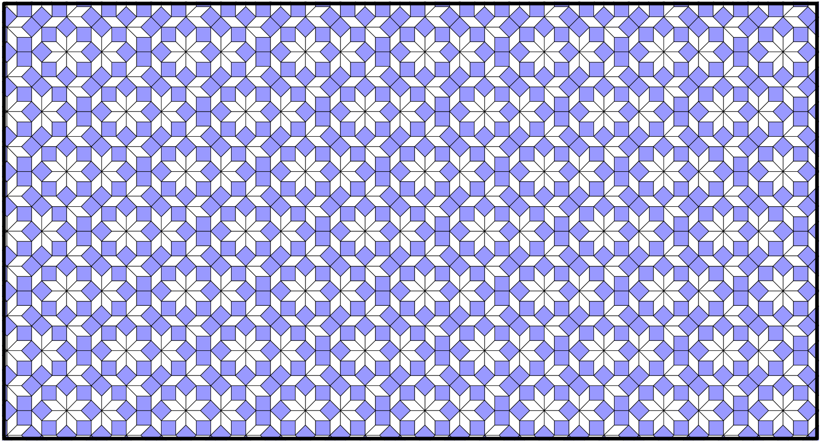

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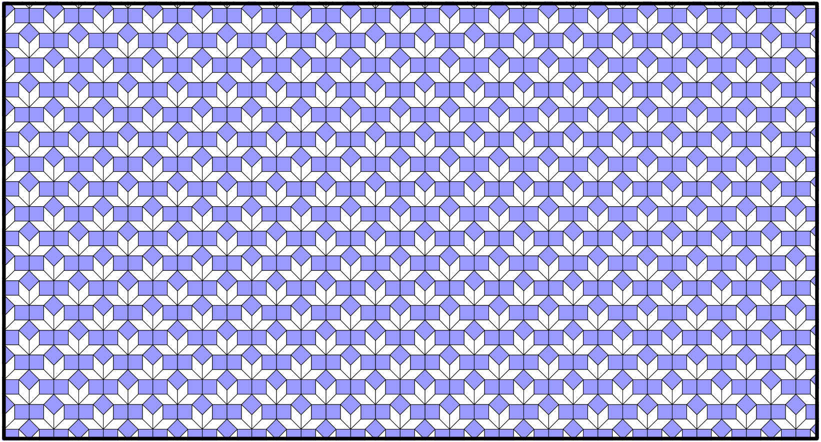

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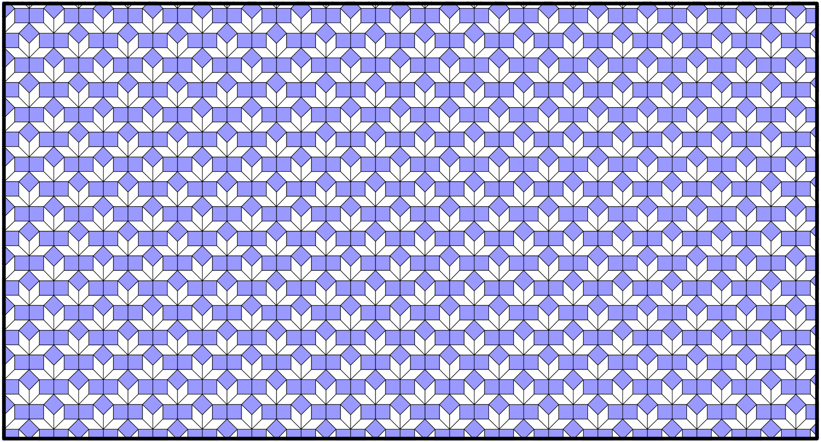

The theorem does not apply, but planarity is nevertheless enforced!

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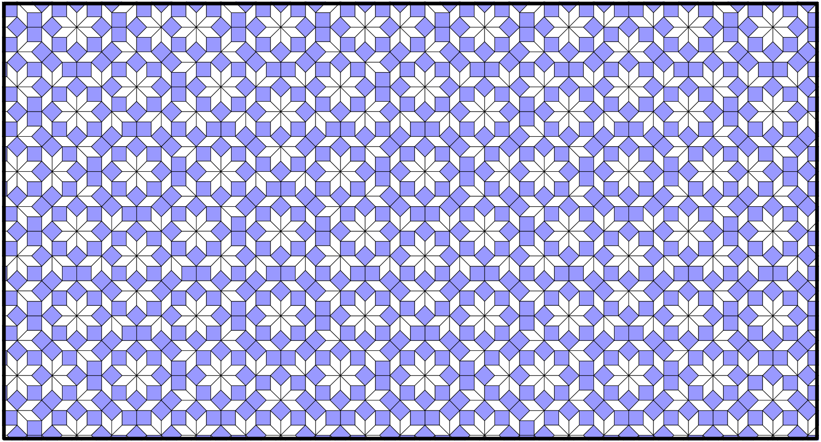

Moreover, AB tilings are those maximizing the rhombus frequencies.

Thank you for your attention!