(Projective) Subdynamics of Multidimensional Subshifts, part II.

SubTile 2013

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Summary

What happened yesterday (between 15:00 and 16:00) ?

- Difficulty to characterize soficness in higher dimension
- Projective subdynamics and subactions of sofic subshifts ?
- Hochman's result

Theorem (Hochman 2008)

- Any effective $\mathbb{Z}^d$-subshift may be obtained as the subaction of a $\mathbb{Z}^{d+2}$ sofic subshift.
- Any effective $\mathbb{Z}^d$ dynamical system may be obtained as the subaction of a $\mathbb{Z}^{d+2}$ sofic subshift.

But before that...

Let's go back to slide 18

Conjecture (Jeandel)

X is sofic $\Leftrightarrow$ $X^{\mathbb{Z}}$ is sofic.

But before that...

Let's go back to slide 18

Conjecture (Jeandel)

X is sofic $\not\Leftarrow$ $X^{\mathbb{Z}}$ is sofic.

There might be a conter-example based on quasi-sturmian words !

$\rightsquigarrow$ see M. Sablik's talk.

Outline

1. Effective subshifts as projective subdynamics of sofic subshifts
   - Substitutive subshifts
   - Hochman's proof
2. From $d+2$ to $d+1$
   - A four layers construction
   - Computation stripes
   - Communication channels
3. Projective subdynamics of SFT
   - Stability and unstability
   - Pavlov ans Schraudner's classification
   - Projective subdynamics of strongly irreducible SFT

Substitutive subshifts

We consider only *rectangular substitutions* on a finite alphabet A.

If s is such a substitution, the *s-patterns* are the $s^n(a)$ for every letter a and every integer $n \in \mathbb{N}$ (if they are well-defined).

Definition

Let s be a rectangular substitution on A. Then the *substitutive subshift generated by s* is

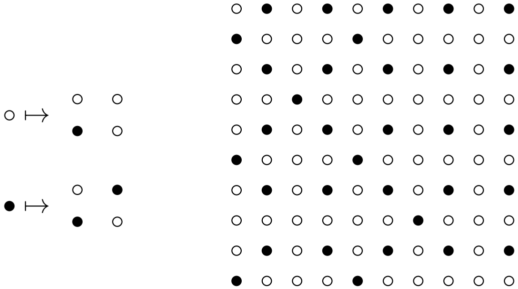
$$X_s = \left\{ x \in A^{\mathbb{Z}^2} : \text{ every pattern of } x \text{ is a } s\text{-pattern} \right\}.$$

Mozes' Theorem

Theorem (Mozes, 1989)

If the substitution s has *good properties* (for instance deterministic), then the subshift X_s is sofic.

Idea of the proof for 2×2 substitutions



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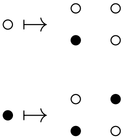

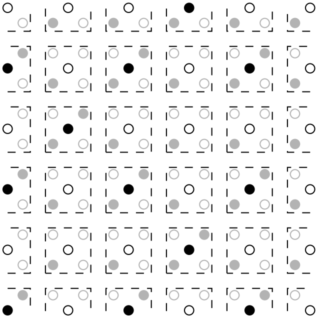

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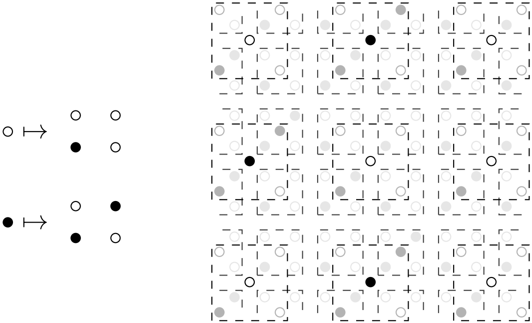


Hochman's proof: a 3D construction

Start with two rectangular substitutions s_3 and s_5

$$s_3 \quad \bullet \mapsto \begin{array}{ccc} \circ & \circ & \bullet \\ \circ & \bullet & \circ \\ \bullet & \circ & \circ \end{array}, \quad \circ \mapsto \begin{array}{ccc} \circ & \circ & \circ \\ \circ & \bullet & \circ \\ \bullet & \circ & \circ \end{array}$$

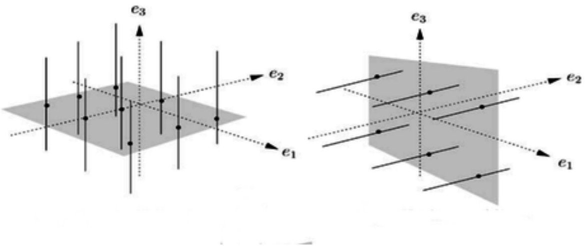
$$s_5 \quad \bullet \mapsto \begin{array}{ccccc} \circ & \circ & \circ & \circ & \bullet \\ \circ & \circ & \circ & \bullet & \circ \\ \circ & \circ & \bullet & \circ & \circ \\ \circ & \bullet & \circ & \circ & \circ \\ \bullet & \circ & \circ & \circ & \circ \end{array}, \quad \circ \mapsto \begin{array}{ccccc} \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \bullet & \circ \\ \circ & \circ & \bullet & \circ & \circ \\ \circ & \bullet & \circ & \circ & \circ \\ \bullet & \circ & \circ & \circ & \circ \end{array}$$

Mozes' result $\Rightarrow$ 2D *sofic subshifts* W_3 and W_5.

Hochman's proof: a 3D construction

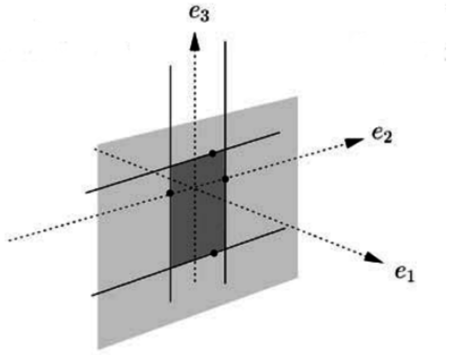
Identical copies of W_3 along direction $\vec{e_3}$ and of W_5 along $\vec{e_2}$

- Copies of W_3 produce *vertical lines*
- Copies of W_3 produce *horizontal lines*



Hochman's proof: a 3D construction

Thus some rectangles appear !



And all rectangles are the same on one plane.

Hochman's proof: a 3D construction

These rectangles have good properties

- there are only finitely many planes with infinite rectangles
- each set $[k, k+n]\vec{e_2}$ will appear in arbitrarily large rectangles

Thus if $\mathcal{M}$ is a TM that enumerates F

- we can put calculations of $\mathcal{M}$ (real time Turing machine) in each rectangle
- each time a forbidden pattern is produced, its presence is checked inside the rectangle
- rectangles repartition $\Rightarrow \mathbb{Z}\vec{e_2}$ is entirely scanned

$\Rightarrow$ The subshift X_F exactly appears on $\mathbb{Z}\vec{e_2}$.

Outline

From $d+2$ to $d+1$

Hochman's result for effective subshifts can be made *optimal* in terms of dimension.
(since there exist non-sofic effective subshifts, dimension d is impossible)

Theorem (Durand, Romaschenko & Shen 2011, A.& Sablik 2013)

Any effective $\mathbb{Z}^d$-subshift may be obtained as the projective subdynamics of a $\mathbb{Z}^{d+1}$ sofic subshift.

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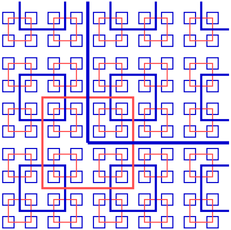
Two independent proofs

- the first one is based on *self-similar tilings*
- the second one uses *Robinson like* techniques

From $d+2$ to $d+1$: Sketch of the proof

What about Robinson tiling ?



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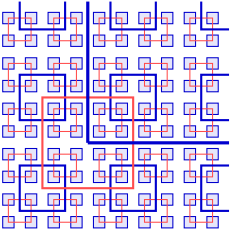

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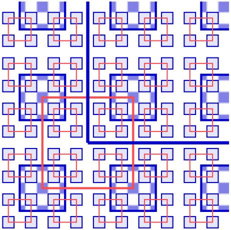

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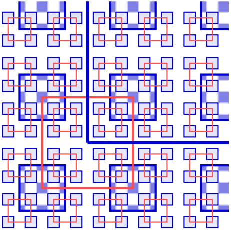

But...

- Computation zones are squares !
- How to solve the *disconnected tape* problem ?

A four layers construction

How to realize an effective 1D-subshift $\Sigma \subset \mathcal{A}_\Sigma^{\mathbb{Z}}$ as PS of a 2D sofic subshift ?

- SFT made of four layers
  - first layer: configuration $x \in \mathcal{A}_\Sigma^{\mathbb{Z}}$ that will be checked
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  - fourth layer: TM $\mathcal{M}_{\mathtt{Search}}$ that helps the TM $\mathcal{M}_{\mathtt{F}}$ to scan entirely x
- all layers but the first are finally erased with a letter-to-letter block map

$x \in \mathcal{A}_\Sigma^{\mathbb{Z}}$

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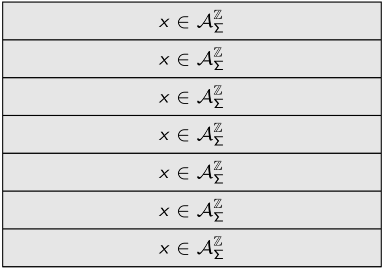

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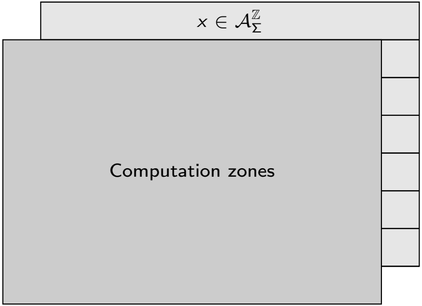

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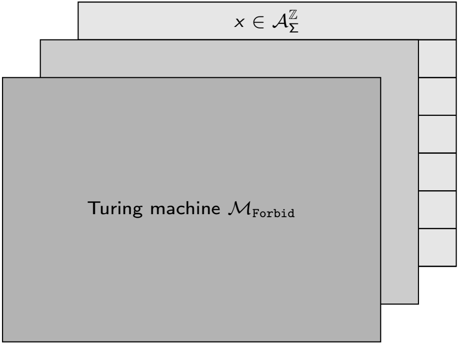

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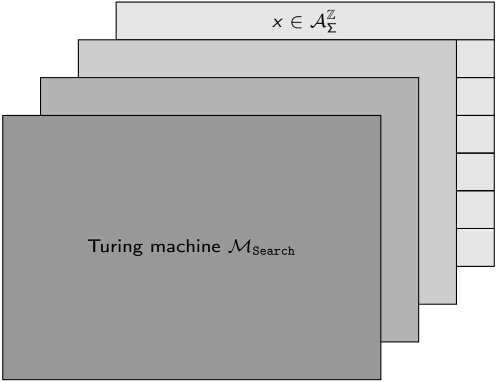

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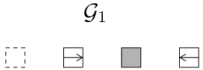
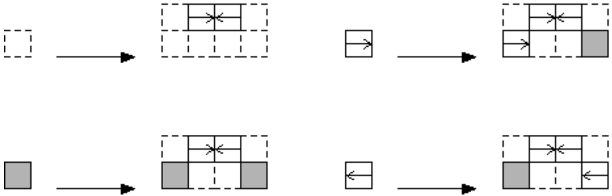
Layer 2: Computation zones

Alphabet $\mathcal{G}_1$

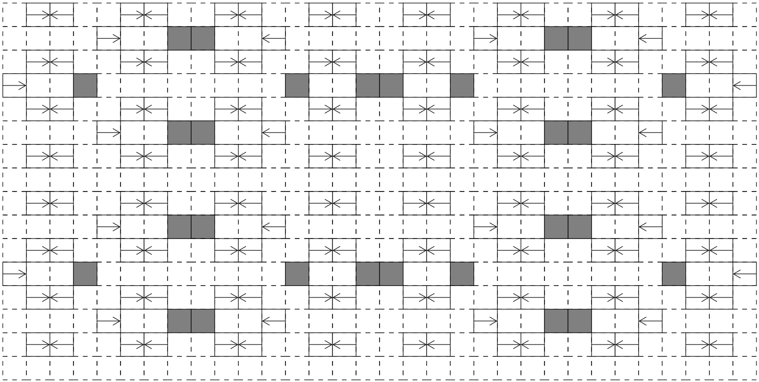


Substitution rules of $s_{\texttt{Grid}}$:



Layer 2: Computation zones

After some iterations...



□ : communication tile

⇥, ⇤, ■ : computation tiles

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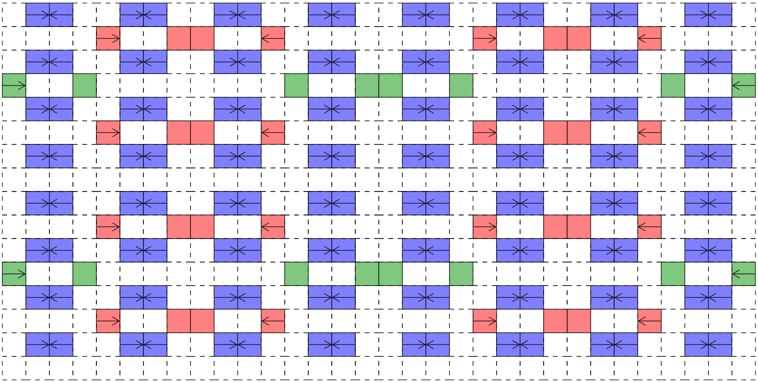

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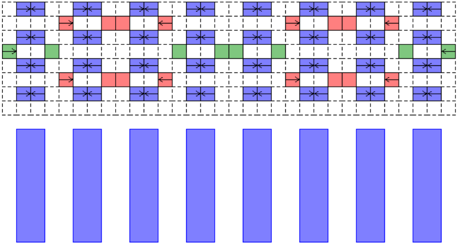
Layer 2: Computation zones

Stripes of different levels (level 1, level 2, level 3):



A stripe of level n has the following properties

- width 2^n,
- one line of computation every 2^n lines.

Layer 2: Computation zones

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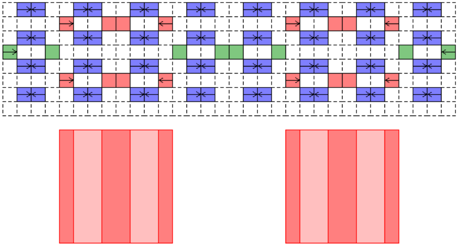

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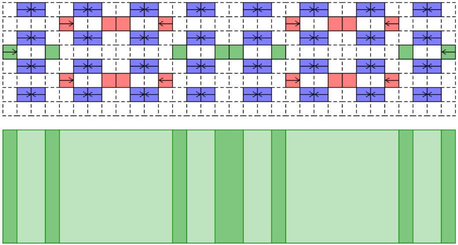

A stripe of level n has the following properties

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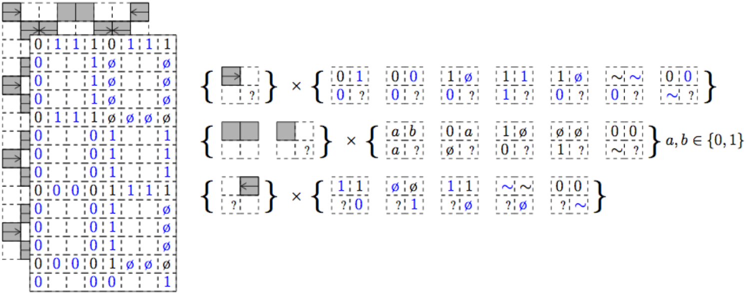
Layer 2: the clock

To initialize calculations we code a clock by local rules



In a level n stripe, calculations are initialized every 2^{2^n} steps of calculation.

Layer 3: How to detect forbidden patterns ?

- $\mathcal{M}_{\texttt{Forbid}}$ generates forbidden patterns of Σ

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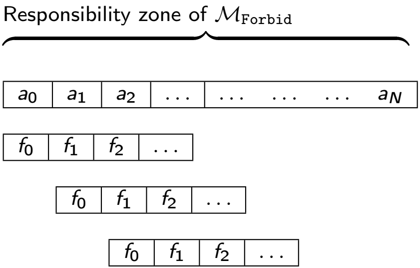
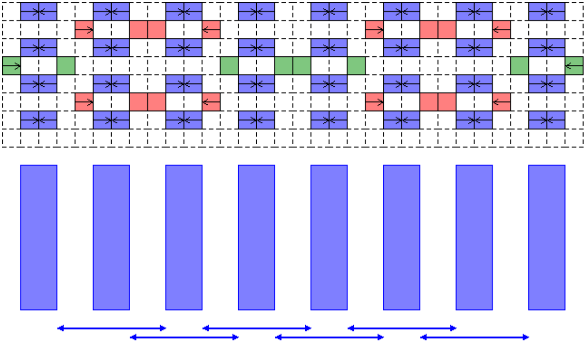


- to get symbol a_k from level 1, $\mathcal{M}_{\texttt{Forbid}}$ is helped by $\mathcal{M}_{\texttt{Search}}$: $\mathcal{M}_{\texttt{Forbid}}$ gives the address k and gets a_k.

Responsibility zone of $\mathcal{M}_{\texttt{Forbid}}$

Responsibility zones must overlap



A Turing machine $\mathcal{M}_{\texttt{Forbid}}$ of level n may ask help from a $\mathcal{M}_{\texttt{Search}}$ of same level or an adjacent $\mathcal{M}_{\texttt{Search}}$ of same level.

Responsibility zone of $\mathcal{M}_{\texttt{Forbid}}$

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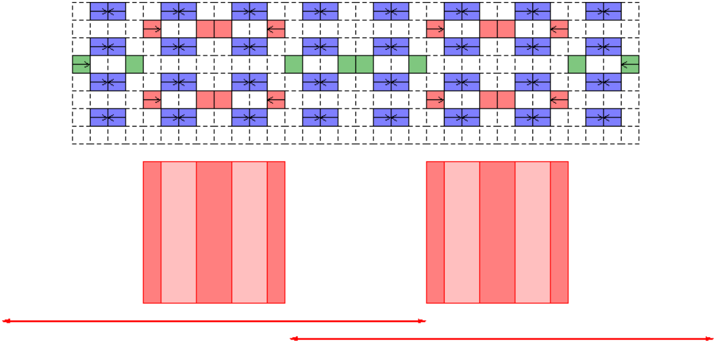

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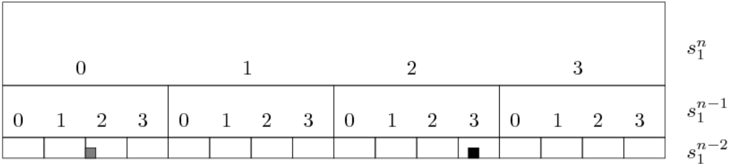
Layer 4 : Turing machine $\mathcal{M}_{\texttt{Search}}$

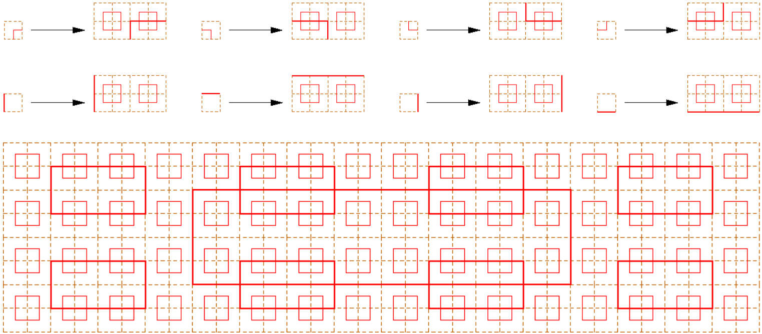
A $\mathcal{M}_{\texttt{Search}}$ machine of level n can communicate with $\mathcal{M}_{\texttt{Search}}$ machines of levels $n-1$ and $n+1$.
Given a computation stripe of level n, each symbol is given an address, and this address is compatible with addresses of levels $n-1$ and $n+1$.



The address of ■ is 231 and the address of ■ is 020.

Communication between $\mathcal{M}_{\mathtt{Search}}$ of different levels

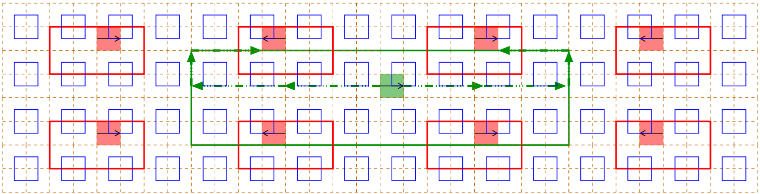
With a new alphabet $\mathcal{G}_2$, we construct *communication channels*



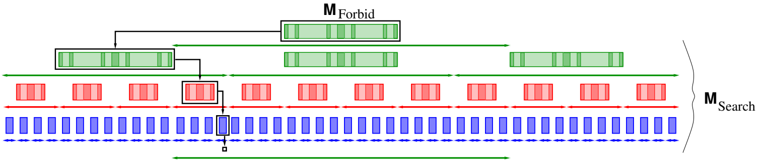
Communication between $\mathcal{M}_{\texttt{Search}}$ of different levels

Communication channels are such that

- every tile ⇾ or ⇽ is in the center of a rectangle of level n;
- every rectangle of level n is connected to the ⇾ and ⇽ of two stripes of level $n-1$



$\mathcal{M}_{\texttt{Search}}$ works !



The machines $\mathcal{M}_{\texttt{Search}}$ work as we expect:

- every $\mathcal{M}_{\texttt{Search}}$ has enough space to code addresses
- every $\mathcal{M}_{\texttt{Search}}$ has enough time to perform calculations (exponential clock)

From $d + 2$ to $d + 1$: Sketch of the proof

A four layers construction How to realize an effective 1D-subshift $\Sigma \subset 1^{\mathbb{Z}}$ as PS of a 2D sofic subshift ?

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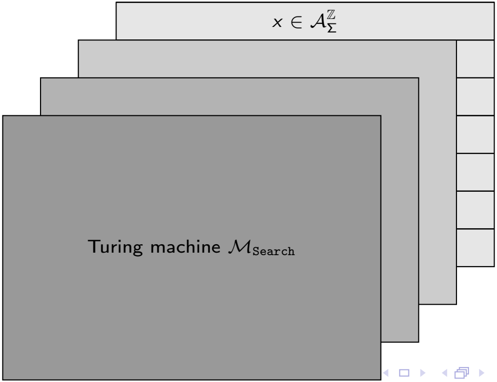

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Some applications

- Characterization of possible entropies of 2D SFT [Hochman & Meyerovitch, 2010]
- Multidimensional effective S-adic subshifts are sofic [A. & Sablik, submitted]
- There exists a sofic subshift whose quasi-periodic configurations have a non-recursively bounded periodicity function [Ballier & Jeandel, 2010]
- A computable planar tiling admits local rules [Fernique & Sablik, 2012]

Improvement, Limitation and Question

- Is it possible to determinize the construction (deterministic SFT) ?
  $\rightsquigarrow$ It should be... [Guillon & Zinoviadis, in progress]

- The construction is highly constrained, in the sense that the sofic subshift is constant along the vertical direction ($\Rightarrow$ zero entropy).
  $\rightsquigarrow$ What are PS of mixing sofic subshifts/SFT ?

- Is it possible to obtain any 1D effective dynamical system as a subaction of a 2D sofic subshift ?
  $\rightsquigarrow$ No, a conter-example is the mirror dynamical system

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1D Sofic subshifts as projective subdynamics of 2D SFT

The *limit set* of a cellular automaton f is

$$\Lambda(f) = \bigcap_{n\in\mathbb{N}} f^n(A^{\mathbb{Z}^d}).$$

This is always a subshift, which can be seen as

- the set of configurations that can be reached after arbitrarily many iterations,
- the biggest set on which f is surjective.

$\Lambda(f)$ is *stable* if the intersection is finite, *unstable* otherwise.

Natural question: which subshifts can arise as stable/unstable limit sets of CA ?

Stability and unstability

We can approximate globally admissible configurations by locally admissible ones.

$$L^{\delta,n} := L + [-n\vec{1}; +n\vec{1}]$$

$$X_{L,n} := \left\{ x|_L : x \in \mathcal{A}^{L^{\delta,n}} \wedge \forall F \subsetneq L^{\delta,n} \text{ finite: } x|_F \notin \mathcal{F} \right\}$$

$$\text{Then one has } P_L(X) = \bigcap_{n \geq 0} X_{L,n}$$

Definition

Given X a $\mathbb{Z}^d$-subshift and $L \lneq \mathbb{Z}^d$ a k-dimensional sublattice

- $P_L(X)$ is *stable* if $\exists N \in \mathbb{N}, \forall n \geq N$: $X_{L,n} = X_{L,N} = P_L(L)$.
- $P_L(X)$ is *unstable* if $\forall n \in \mathbb{N}, \exists n \geq N$: $X_{L,n} \subsetneq X_{L,N}$.

1D Sofic subshifts as projective subdynamics of 2D SFT

Classification in [Pavlov & Schraudner, preprint] based on the notions of

- *Universal Periods (UP)* $\approx$ all configurations are periodic (if you forget a bounded finite numbers of points).
- *Good sets of periods (GSP)* $\approx$ you have enough periodic configurations to know where you are in a graph presentation.

<table>
<tr><td colspan="4"></td><td>Stable</td><td>Unstable</td></tr>
<tr><td colspan="4">SFT</td><td>✓</td><td>×</td></tr>
<tr><td rowspan="4">Stricly sofic</td><td colspan="3">$h > 0$</td><td>✓</td><td>✓</td></tr>
<tr><td rowspan="3">$h = 0$</td><td colspan="2">UP</td><td>×</td><td>×</td></tr>
<tr><td rowspan="2">no UP</td><td>GSP</td><td>✓</td><td>✓</td></tr>
<tr><td>no GSP</td><td>×</td><td>✓</td></tr>
</table>

What about non sofic subshifts ?

Theorem (Guillon, 2011)

Every $\mathbb{Z}$-effective subshift that contains a sofic subshift of positive entropy is the projective subdynamics of some $\mathbb{Z}^2$-SFT.

Theorem (Sablik & Schraudner, *in progress*)

A certain class of $\mathbb{Z}$-effective subshift that contains a subshift of positive entropy is the subdynamics of some $\mathbb{Z}^2$-SFT.

Mixing subshift and strong irreducibility

Definition

A subshift $X \subset A^{\mathbb{Z}^d}$ is *mixing* if

$$\forall U, W \subsetneq \mathbb{Z}^d \text{ finite, disjoint, non-empty}, \exists M_{U,V} \in \mathbb{N}^* \text{ s.t.}$$

$$\forall \vec{v} \in \mathbb{Z}^d \text{ s.t. } d(U, \vec{v} + W) > M$$

$$\forall y, z \in X \Rightarrow \exists x \in X \text{ s.t. } x|_U = y|_U \text{ and } x|_{\vec{v}+W} = z|_{\vec{v}+W}.$$

Definition

A subshift $X \subset A^{\mathbb{Z}^d}$ is *strongly irreducible* if there exists *a gap* $g \in \mathbb{N}^*$ s.t.

$$\forall U, V \subsetneq \mathbb{Z}^d \text{ finite, disjoint, non-empty and } d(U, V) > g,$$

$$\forall y, z \in X \Rightarrow \exists x \in X \text{ s.t. } x|_U = y|_U \text{ and } x|_V = z|_V.$$

Mixing sofic subshift as PS of strongly irreducible SFT

In 2D it is possible to define other mixing properties

- block gluing (sets are rectangles)
- corner gluing
- uniform filling property

Mixing sofic subshift as PS of strongly irreducible SFT

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Schraudner, preprint

Any 1D mixing sofic subshift is the stable PS of a strongly irreducible 2D SFT.

Conclusion

- 1D effective subshifts as PS of 2D sofic subshifts
- Classification of 1D sofic subshift that are PS of 2D SFT
- Another approach: impose that lines are in some subshift X_H, what subshift X_V can you get on the columns ?

Conclusion

- 1D effective subshifts as PS of 2D sofic subshifts
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Thank you for your attention !