(Projective) Subdynamics of Multidimensional Subshifts, part I.

SubTile 2013

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Outline

Full-shift, shift action and subshift

- $\mathcal{A}$ a finite alphabet and $d \in \mathbb{N}$
- $x \in \mathcal{A}^{\mathbb{Z}^d}$ is a *configuration*
- $\mathcal{A}^{\mathbb{Z}^d}$ endowed with the prodiscrete topology is a compact metric space
- *shift action* $\sigma : \mathbb{Z}^d \times \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$, $\left(\sigma_{(n_1,\ldots,n_d)}(x)\right)_{(i_1,\ldots,i_d)} = x_{(i_1+n_1,\ldots,i_d+n_d)}$
- the dynamical system $\left(\mathcal{A}^{\mathbb{Z}^d}, \sigma\right)$ is the *d-dimensional full-shift on $\mathcal{A}$*

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Definition

A *subshift* is a closed and σ-invariant subset of $\mathcal{A}^{\mathbb{Z}^d}$.

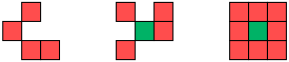
$\left\{x \in \{0,1\}^{\mathbb{Z}^2} : x_{(i,j)} = 1 \Leftrightarrow i = j = 0\right\}$ not σ-invariant !

$\left\{x \in \{0,1\}^{\mathbb{Z}^2} : \text{ only one 1 appears in } x\right\}$ not closed !

$\left\{x \in \{0,1\}^{\mathbb{Z}^2} : \text{ at most one 1 appears in } x\right\}$ is a subshift.

Combinatorial point of view

- A *pattern* is a local function $p : S \to \mathcal{A}$, where $S \subset \mathbb{Z}^d$ is finite.



- Given a pattern $u \in \mathcal{A}^S$, it generates the *cylinder*

$$[u] = \left\{ x \in \mathcal{A}^{Z^{\mathbf{d}}} : x|_S = u \right\}.$$

- If F is a set of patterns, the *subshift generated by F* is

$$X_F = \left\{ x \in \mathcal{A}^{Z^{\mathbf{d}}} : \text{ no pattern of } F \text{ appears in } x \right\}.$$

- A subshift is thus the complement of a union of cylinders

$$X_F = \mathcal{A}^{\mathbb{Z}^{\mathbf{d}}} \setminus \left(\bigcup_{\mathbf{i} \in \mathbb{Z}^{\mathbf{d}}, u \in F} \sigma_{\mathbf{i}}([u]) \right).$$

Language of a subshift

Definition

The *language of size n* of a $\mathbb{Z}^d$-subshift X is

$$\mathcal{L}_n(X) := \left\{p : [-n;n]^d \to \mathcal{A} : \exists x \in X, p \text{ appears in } x\right\}.$$

The *language* of a $\mathbb{Z}^d$-subshift X is

$$\mathcal{L}(X) := \bigcup_{n \geq 0} \mathcal{L}_n(X).$$

The *complement of the language* $\mathcal{L}(X)^c$ is the biggest set of forbidden patterns.

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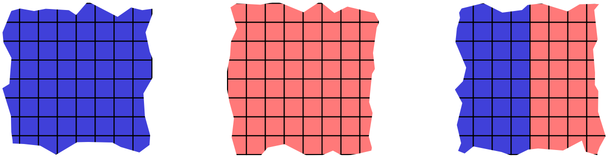
Proposition

The topological and combinatorial definitions coincide.

Subshifts of finite type

The subshift $X_{\{}$  $_{\}}$ contains the following configurations



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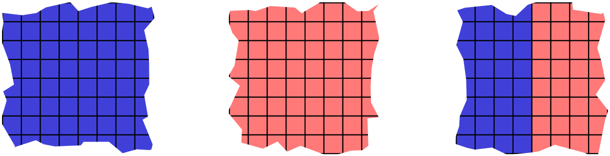


Definition

A subshift is *of finite type (SFT)* if it can be defined by a finite set of forbidden patterns. It is of *rank k* if these finite patterns may be chosen of size k.

- simplest class for the combinatorial definition
- 2D-SFT $\equiv$ tilings by Wang tiles
- closely related to cellular automata theory

Sofic subshifts

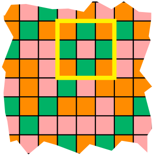
Definition

A *sofic subshift* is the image of a SFT under a continuous and σ-commuting map.

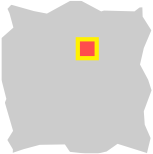
continuous and σ-commuting map $\Leftrightarrow$ Sliding block map (cellular automaton) [Hedlund, 1969]

$$\Phi : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{B}^{\mathbb{Z}^d} \text{ given by the local function } \phi$$

$x \in \mathcal{A}^{\mathbb{Z}^2}$



$\Phi(x) \in \mathcal{B}^{\mathbb{Z}^2}$



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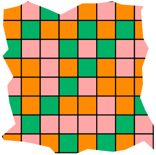

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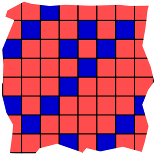

- SFT on which information can be erased.
- On $\mathbb{Z}$, sofic subshifts are exactly those recognized by finite automata.
- In higher dimension, no characterization is known.

An example of purely sofic subshift

$$\text{Let } X_{\leq 1} = \left\{x \in \{0,1\}^{\mathbb{Z}^2} : \text{ at most one 1 appears in } x\right\}.$$

- Suppose that $X_{\leq 1}$ is a rank k SFT.
- Then a configuration that contains two 1's at distance $2k+1$ cannot be rejected.

$$\Rightarrow X_{\leq 1} \text{ is not an SFT!}$$

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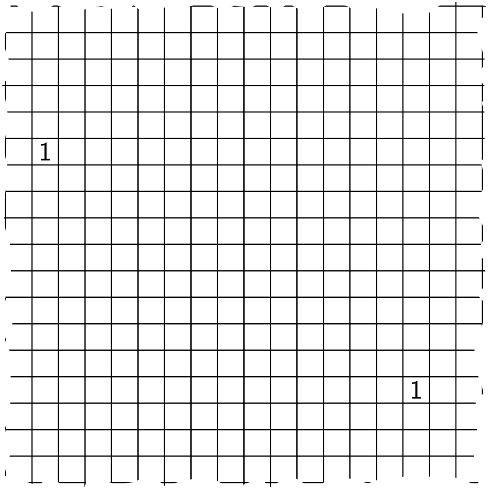

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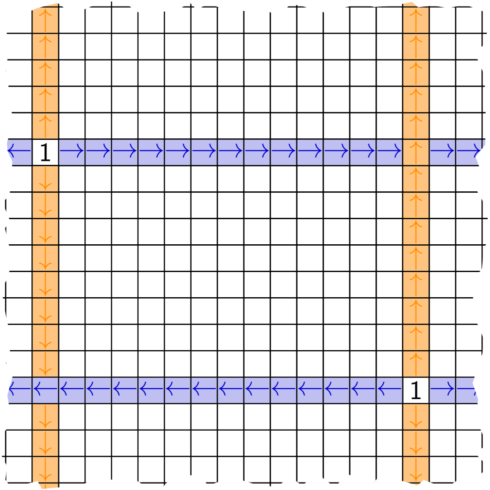

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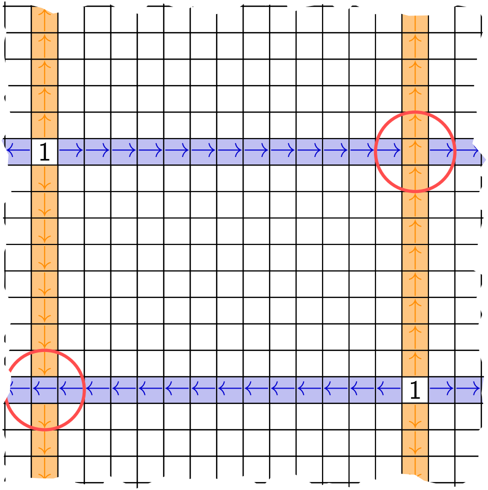
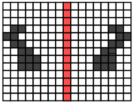


An example of non-sofic subshift

The *mirror subshift* is defined on alphabet { □ , ■ , red square } by

$$X_{\text{mirror}} = \{ \square , \blacksquare \}^{\mathbb{Z}^2} \bigcup \left\{ \right.$$



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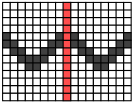

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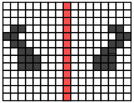
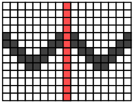
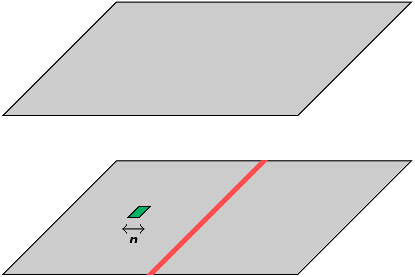
$$X_{\text{mirror}} = \{ \square , \blacksquare \}^{\mathbb{Z}^2} \bigcup \Big\{$$  $,$  $, \dots \Big\}$

Suppose X_{mirror} is sofic.
Then $\exists \Sigma \subset A^{\mathbb{Z}^2}$ a k-SFT and Π a block map of order r, such that

$$\Pi : \Sigma \to X_{\text{mirror}} \text{ is onto.}$$



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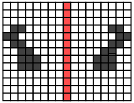
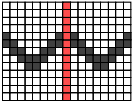
$X_{\text{mirror}} = \{ □ , ■ \}^{\mathbb{Z}^2} \bigcup \Big\{$  ,  $, \dots \Big\}$

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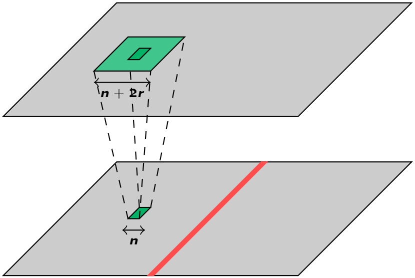

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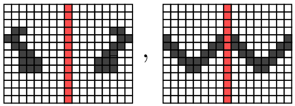

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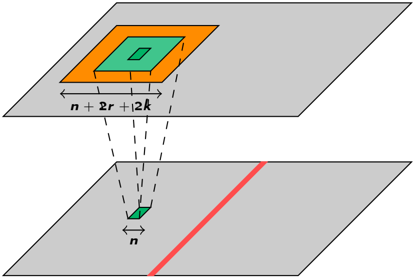

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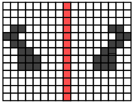
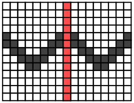
$X_{\text{mirror}} = \{ □ , ■ \}^{\mathbb{Z}^2} \bigcup \{$  ,  , $\dots \}$

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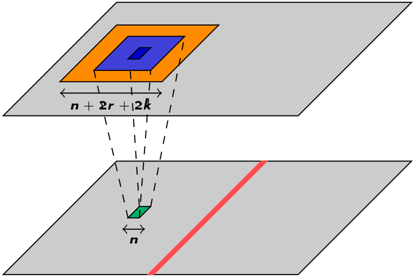

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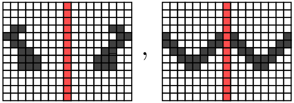

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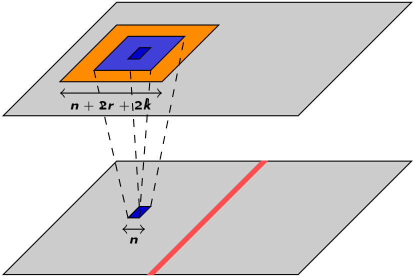

Effectively closed subshifts

$$\text{SFT} \subsetneq \text{Sofic susbhifts} \subsetneq \textit{Effectively closed}$$

Definition

A subshift is *effectively closed* (or *effective*) if its complement is a computable union of cylinders.

Property

X is effectively closed if and only one of the followings holds

(i) $X = X_{\mathcal{F}}$ for some recursively enumerable set $\mathcal{F}$ of forbbiden patterns

(ii) $X = X_{\mathcal{F}}$ for some recursive set $\mathcal{F}$ of forbbiden patterns

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Remark: There exist non effectively closed subshifts (countability argument).

Turing machines and SFT (I)

A *Turing machine* is a tuple $\mathcal{M} = (Q, \Gamma, \sharp, q_0, \delta, Q_F)$ where:

- Q is a finite set of states, $q_0 \in Q$ is the initial state;
- Γ is a finite alphabet;
- $\sharp \notin \Gamma$ blank symbol
- $\delta : Q \times \Gamma \to Q \times \Gamma \times \{\leftarrow, \downarrow, \to\}$ transition function;
- $F \subset Q_F$ finite set of final states.

The rule $\delta(q_1, x) = (q_2, y, \leftarrow)$ will be encoded by the pattern

$z \leftarrow q_2$	y	z'
z	$x \leftarrow q_1$	z'

Turing machines and SFT (II)

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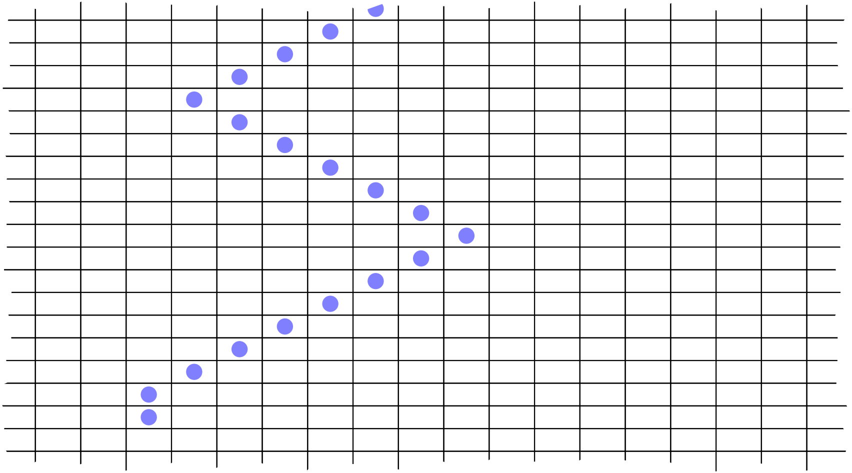

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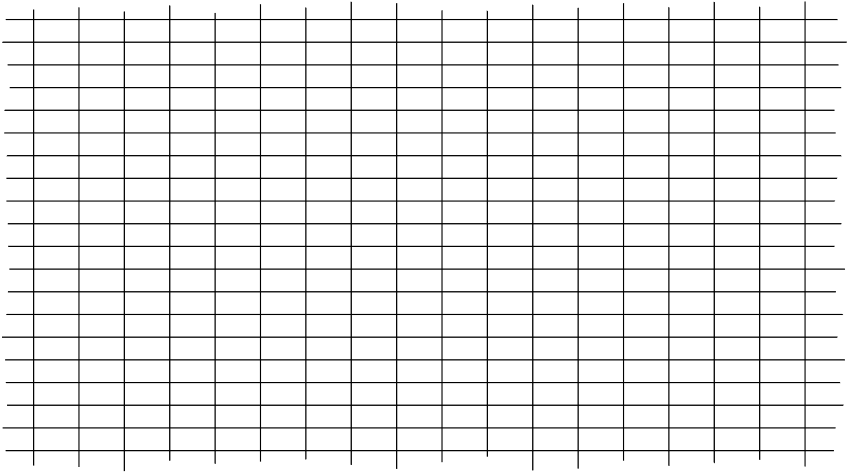

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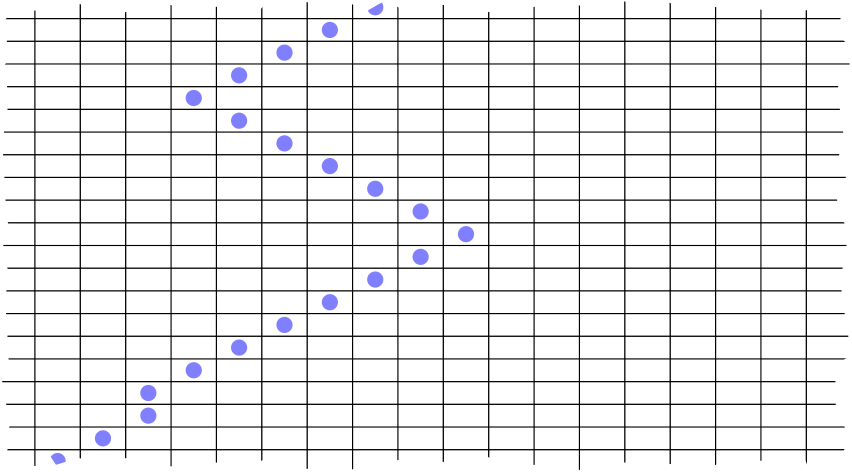

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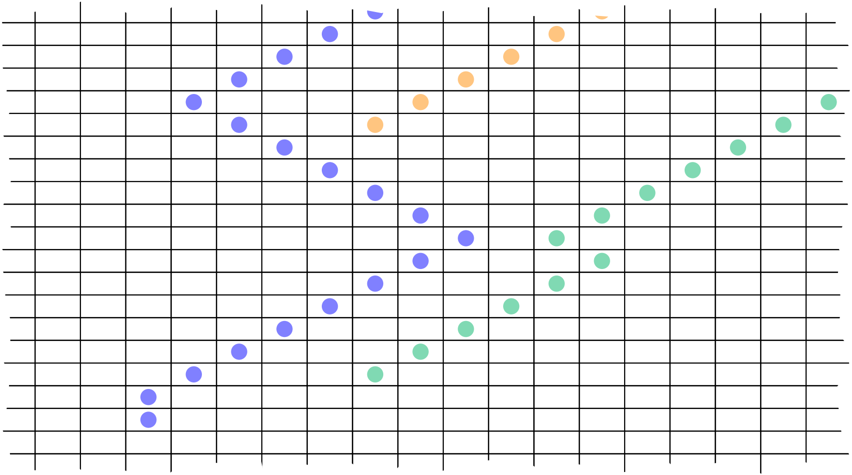

2D vs 1D sofic subshifts

1D sofic subshifts

- $X_F = \emptyset$? is decidable
- entropy is computable (nonnegative rational multiples of log of Perron numbers)
- representation by finite automata/matrix
- every SFT has a periodic configuration
- soficness $\Leftrightarrow$ finite number of followers set

2D sofic subshifts

- $X_F = \emptyset$? is undecidable
- entropy is not computable (right recursively enumerable numbers)
- representation by Wang tiles, textile systems
- $\exists$ aperiodic SFT

Necessary conditions for soficness in 2D

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- Too many extender sets implies non-soficness. [Kass & Madden 2013] and [Pavlov, 2013]

Outline

Projective Subdynamics

Initially introduced by Johnson, Kass and Madden in 2007.

Definition

Let $X \subseteq \mathcal{A}^{\mathbb{Z}^d}$ be a $\mathbb{Z}^d$ subshift and $L \lneq \mathbb{Z}^d$ a k-dimensional sublattice $(1 \leq k < d)$. The *L-projective subdynamics of X* is

$$P_L(X) := \{x|_L : x \in L\} \subseteq \mathcal{A}^L.$$

- $\left(P_L(X), \sigma_{L \times P_L(X)}\right)$ is a $\mathbb{Z}^k$-subshift.
- $P_L(X)$: globally admissible configurations of shape L in X.
- Loss of information about the original subshift.

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- $P_L(X)$: globally admissible configurations of shape L in X.
- Loss of information about the original subshift.

In the sequel, we will concentrate on $P_{\vec{e}_1\mathbb{Z}}(X)$ (PS along the horizontal direction).

Entropy and PS

Proposition (Johnson, Kass & Madden, 2007)

$$h_{top}(P_{\vec{e}_1\mathbb{Z}}(X)) \geq h_{top}(X).$$

Proof:

$$\begin{aligned} h_{top}(X) &= \lim_{n\to\infty} \frac{1}{n^2} \log\left(|\mathcal{L}_n(X)|\right) \\ &= \lim_{n\to\infty} \frac{1}{n^2} \log\left(|\mathcal{L}_n(P_{\vec{e}_1\mathbb{Z}}(X))|^n\right) \\ &\leq \lim_{n\to\infty} \frac{1}{n} \log\left(|\mathcal{L}_n(P_{\vec{e}_1\mathbb{Z}}(X))|\right) \\ &= h_{top}(P_{\vec{e}_1\mathbb{Z}}(X)) \end{aligned}$$

Subdynamics

Definition

Let $X \subseteq \mathcal{A}^{\mathbb{Z}^d}$ be a $\mathbb{Z}^d$ subshift and $Y \subseteq B^{\mathbb{Z}^k}$ a $\mathbb{Z}^k$-subshift $(1 \leq k < d)$. Then Y is a *subaction of X* if the dynamical systems $(X, \sigma|_{\mathbb{Z}^k})$ and $(Y, \sigma|_{\mathbb{Z}^k})$ are isomorphic.

- Much stronger than projective subdynamics
- The subshift Y is defined on a possibly non-finite alphabet
- No loss of information

Questions

- What are projective subdynamics of 2D sofic subshifts
- What are projective subdynamics of 2D SFT ?
- What are subactions of sofic subshifts ?
- What are subactions of 2D SFT ?

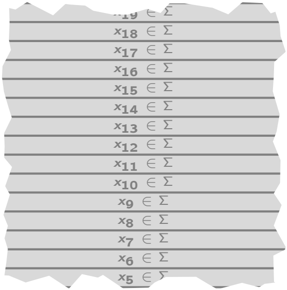
Questions

- What are projective subdynamics of 2D sofic subshifts
  = effective subshifts
- What are projective subdynamics of 2D SFT ?
  ???
- What are 1D subactions of 3D sofic subshifts ?
  = effective dynamical systems
- What are subactions of 2D SFT ?
  ???

What can be PS of sofic subshifts ? (0)

- Trivially, every 1D sofic subshift. . .

SFT $\Sigma^{\mathbb{Z}}$



$X \subset A^{\mathbb{Z}}$ sofic

$\Sigma \subset B^{\mathbb{Z}}$ SFT, $\Pi : \Sigma \to X$ block map

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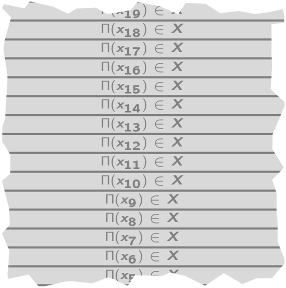
SFT $\Sigma^{\mathbb{Z}}$



$X \subset A^{\mathbb{Z}}$ sofic

$\Sigma \subset B^{\mathbb{Z}}$ SFT, $\Pi : \Sigma \to X$ block map

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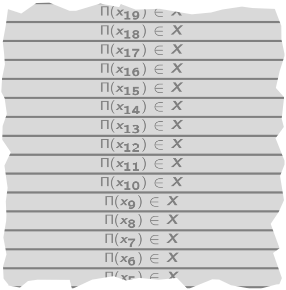

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Conjecture (Jeandel)

X is sofic $\Leftrightarrow$ $X^{\mathbb{Z}}$ is sofic.

What can be PS of sofic subshifts ? (I)

- The 1D subshift $X_{a^n b^n}$.

c	c	c	a	a	a	a	a	b	b	b	b	b	c	c	c	c	a	a	a	a	a	a	a	a	a	b	b	b	c	c

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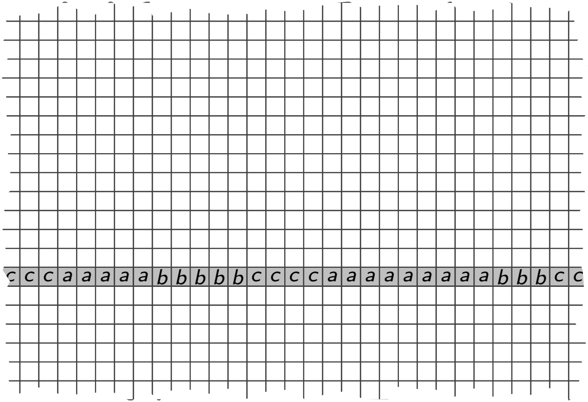

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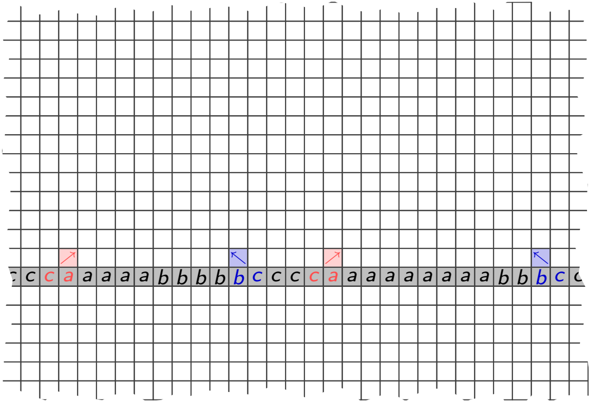

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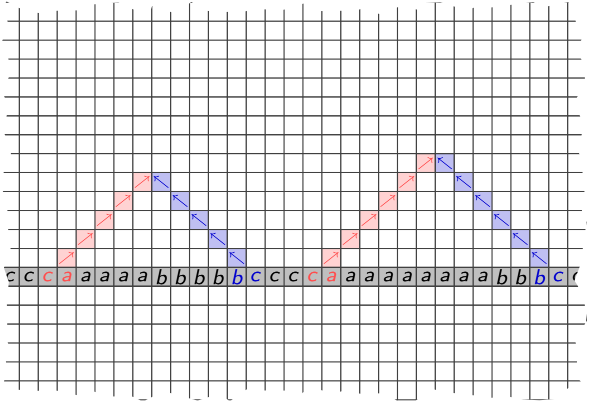

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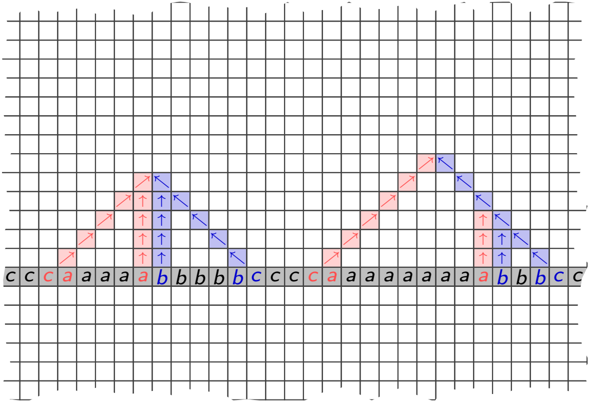

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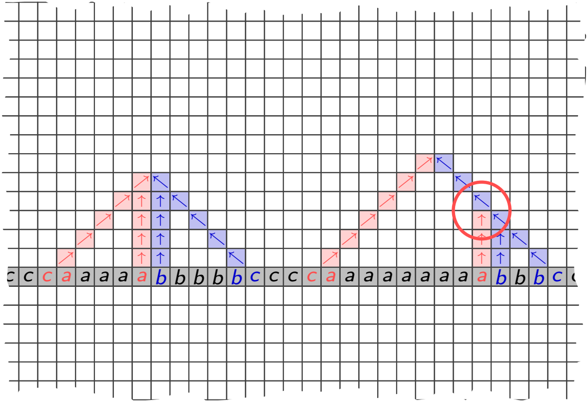

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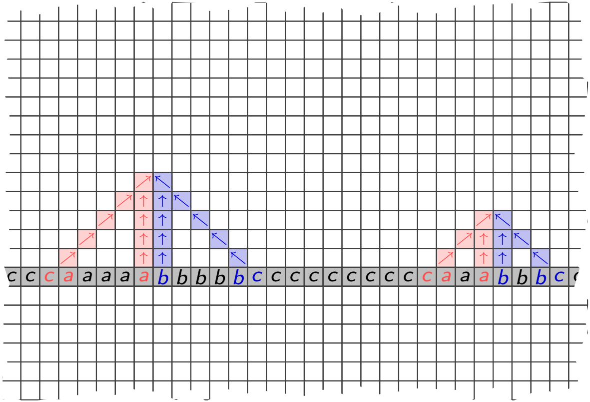

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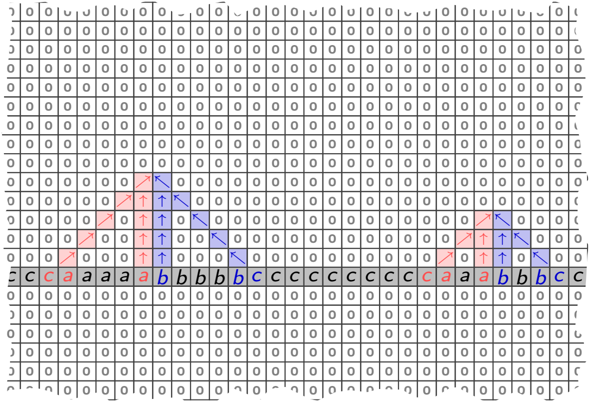

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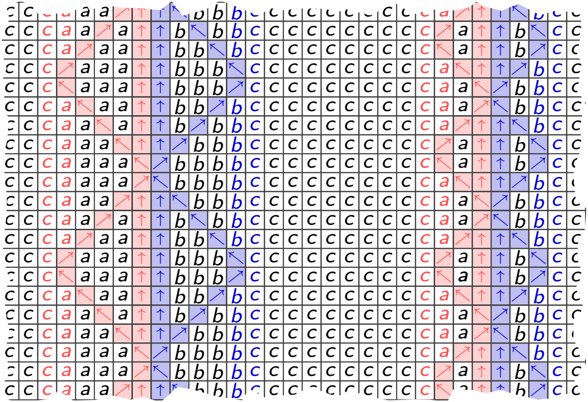
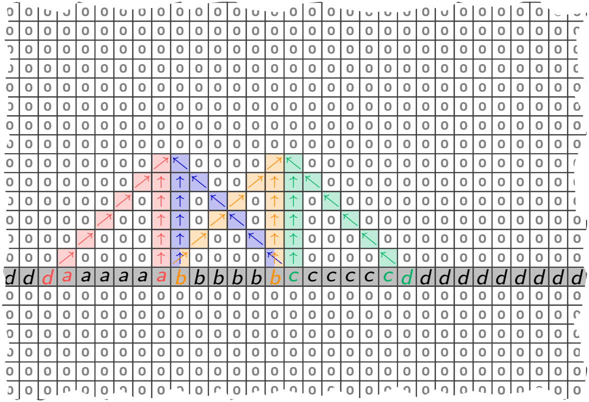


What can be PS of sofic subshifts ? (II)

- The 1D subshift $X_{a^n b^n c^n}$ (neither sofic nor algebraic).



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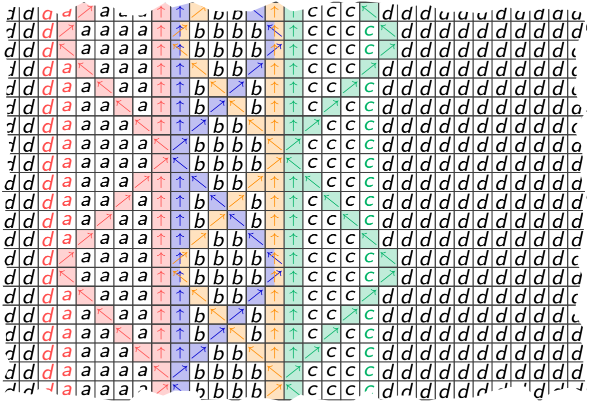

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$\mathcal{A}^{\mathbb{Z}^2}$

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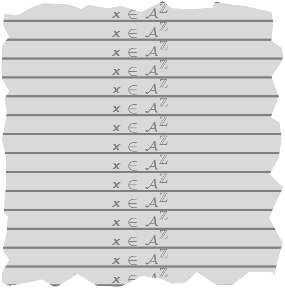
$X_{\left\{ \frac{b}{a} : a,b \in \mathcal{A} \right\}}$
(SFT)



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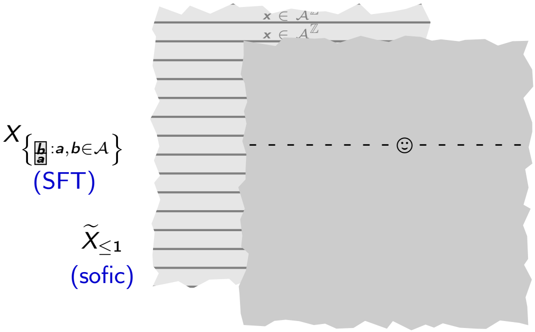

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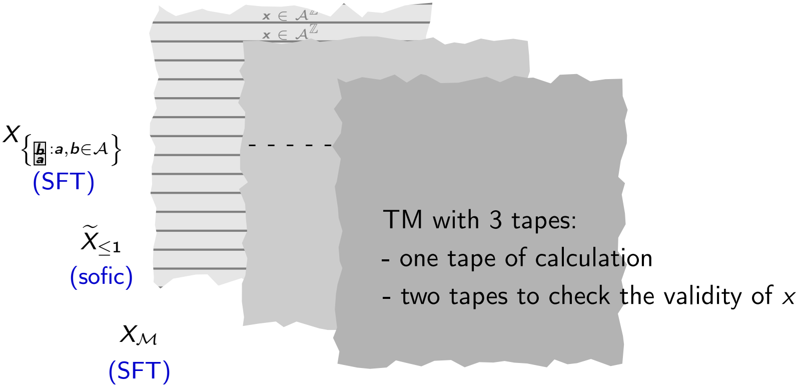

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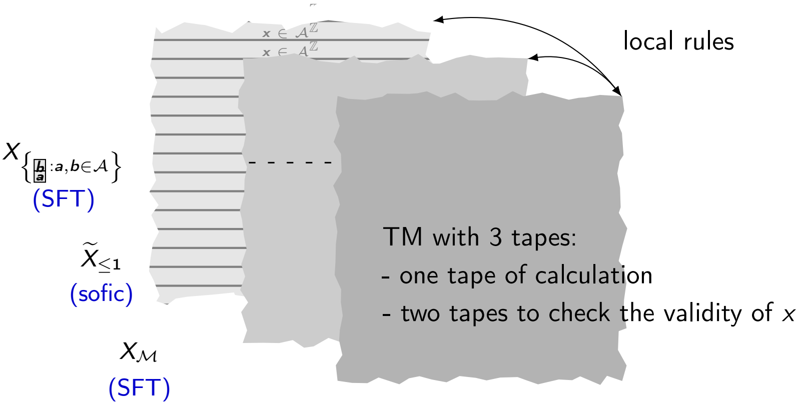

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On the third layer:

- The Turing Machine $\mathcal{M}$ works on the first tape and enumerates forbidden patterns for X (initialization thanks to the second layer).
- Each time a forbidden patterns is produced, it is copied out on the two other tapes.
- Patterns written on the second (resp. third) tape are shifted to the left (resp. right) at each step of computation.

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- If a pattern written on the two last tapes matches with the corresponding pattern in x, then the configuration is forbidden (intercation by local rules with the first layer).
- If a forbidden pattern for X appears in x, it will eventually be detected and the configuration is rejected.
- If $x \in X$, the configuration is accepted.

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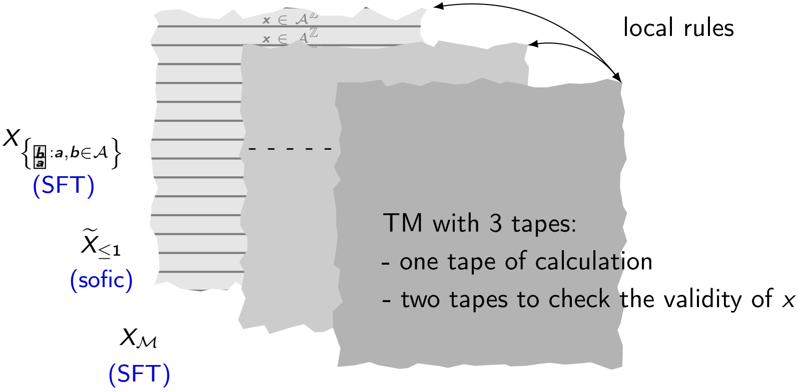

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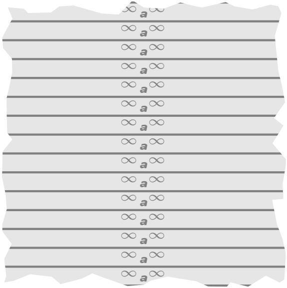

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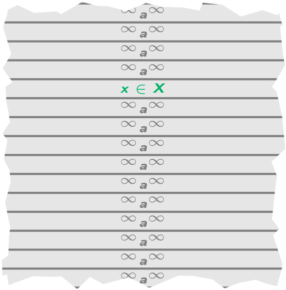

$\Rightarrow X$ is a PS of a sofic subshift

Hochman's result

Theorem (Hochman 2008)

- Any effective $\mathbb{Z}^d$-subshift may be obtained as the subaction of a $\mathbb{Z}^{d+2}$ sofic subshift.
- Any effective $\mathbb{Z}^d$ dynamical system may be obtained as the subaction of a $\mathbb{Z}^{d+2}$ sofic subshift.

The proof is based on

- the use of *Turing machines as SFT*,
- *substitutive tilings* to construct computation zones in 3D.

Conclusion of Part I

- Challenging question: characterize soficness in higher dimension.
- Projective subdynamics and subaction: decrease dimension to better understand 2D subshifts.
- Complete characterization of PS/subactions of sofic subshifts (Hochman)
- Coming soon:
  - Sketch of Hochman's proof...
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Thank you for your attention !