S-adic tiles

Wolfgang Steiner
(joint work with Valérie Berthé and Jörg Thuswaldner)

LIAFA (CNRS, Université Paris Diderot – Paris 7)

SubTile, CIRM, January 17, 2013

S-adic words

Let S be a finite set of substitutions on the alphabet $A = \{1, \ldots, d\}$. An infinite word $\omega \in A^{\mathbb{N}}$ is an *S-adic word* if

$$\omega = \lim_{n \to \infty} \sigma_0 \sigma_1 \cdots \sigma_{n-1}(a_n)$$

for sequences $(\sigma_n)_{n \in \mathbb{N}} \in S^{\mathbb{N}}$ and $(a_n)_{n \in \mathbb{N}} \in A^{\mathbb{N}}$.

S-adic words

Let S be a finite set of substitutions on the alphabet $A = \{1, \dots, d\}$. An infinite word $\omega \in A^{\mathbb{N}}$ is an *S-adic word* if

$$\omega = \lim_{n\to\infty} \sigma_0\sigma_1 \cdots \sigma_{n-1}(a_n)$$

for sequences $(\sigma_n)_{n\in\mathbb{N}} \in S^{\mathbb{N}}$ and $(a_n)_{n\in\mathbb{N}} \in A^{\mathbb{N}}$.

Example

Arnoux-Rauzy sequences on $A = \{1, 2, 3\}$: $S = \{\tau_1, \tau_2, \tau_3\}$,

$$\tau_1 : \begin{array}{ccc} 1 & \mapsto & 1 \\ 2 & \mapsto & 12 \\ 3 & \mapsto & 13 \end{array} \qquad \tau_2 : \begin{array}{ccc} 1 & \mapsto & 21 \\ 2 & \mapsto & 2 \\ 3 & \mapsto & 23 \end{array} \qquad \tau_3 : \begin{array}{ccc} 1 & \mapsto & 31 \\ 2 & \mapsto & 32 \\ 3 & \mapsto & 3 \end{array}$$

e.g.

$$\begin{aligned} \omega &= 1121131121112113112111211311211211311211121131121\cdots \\ &= \tau_1(121312112131211213121213121121312 1\cdots) \\ &= \tau_1\tau_1(2321232123223212321\cdots) \\ &= \tau_1\tau_1\tau_2(3131323131\cdots) \\ &= \tau_1\tau_1\tau_2\tau_3(11211\cdots) \end{aligned}$$

Periodic case (fixed point of a substitution)

If $(\sigma_n)_{n\in\mathbb{N}} \in S^{\mathbb{N}}$ and $(a_n)_{n\in\mathbb{N}} \in A^{\mathbb{N}}$ are periodic with period p, then

$$\omega = \lim_{n\to\infty} (\sigma_0\sigma_1\cdots\sigma_{p-1})^n(a_0)$$

Example

Tribonacci sequence

$$\omega = \lim_{n\to\infty} (\tau_1\tau_2\tau_3)^n(1) = \lim_{n\to\infty} \tau^n(1)$$

$$\tau_1\tau_2\tau_3 : \begin{array}{ccc} 1 & \mapsto & 1213121 \\ 2 & \mapsto & 121312 \\ 3 & \mapsto & 1213 \end{array} \qquad \tau : \begin{array}{ccc} 1 & \mapsto & 12 \\ 2 & \mapsto & 13 \\ 3 & \mapsto & 1 \end{array}$$

$$\tau_1\tau_2\tau_3 = \tau^3$$

Broken line

The *abelianisation map* on A^* is

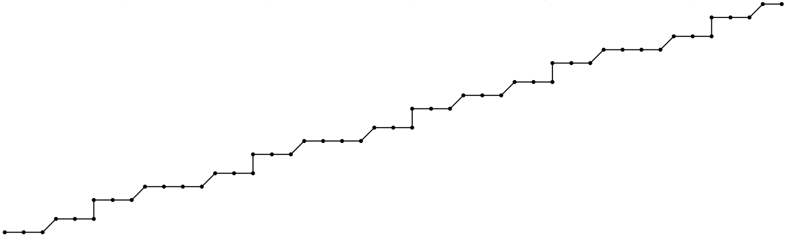
$$\mathbf{l}:\ A^* \to \mathbb{N}^d,\ w \mapsto (|w|_1, |w|_2, \ldots, |w|_d)^t,$$

where $|w|_j$ denotes the number of occurrences of the letter j in w.
The *broken line* associated with $\omega = \omega_0\omega_1 \cdots \in A^{\mathbb{N}}$ has vertex set

$$\{\mathbf{l}(\omega_{[0,n)}) : n \in \mathbb{N}\}, \quad \text{where } \omega_{[0,n)} = \omega_0\omega_1\cdots\omega_{n-1}.$$

Example

$\omega = 1121131121112113112111211311211211311211121131121\cdots$



Rauzy fractal

If the letter frequencies $\lim_{n\to\infty} \frac{|\omega_{[0,n)}|_i}{n}$ exist, let

$$\mathbf{u} = (f_1, f_2, \ldots, f_d)^t$$

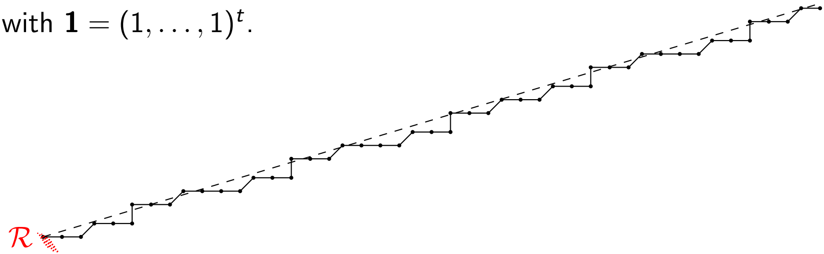
be the *frequency vector* of ω. For $\mathbf{v} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$, let

$$\mathcal{P}(\mathbf{v}) = \{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{x}, \mathbf{v} \rangle = 0\},$$

$\pi_{\mathbf{u},\mathbf{v}}$ the projection onto the hyperplane $\mathcal{P}(\mathbf{v})$ in the direction of $\mathbf{u}$.
The *Rauzy fractal* corresponding to ω is defined by

$$\mathcal{R} = \overline{\{\pi_{\mathbf{u},\mathbf{1}}\, \mathbf{l}(\omega_{[0,n)}) : n \in \mathbb{N}\}} \subset \mathcal{P}(\mathbf{1}),$$

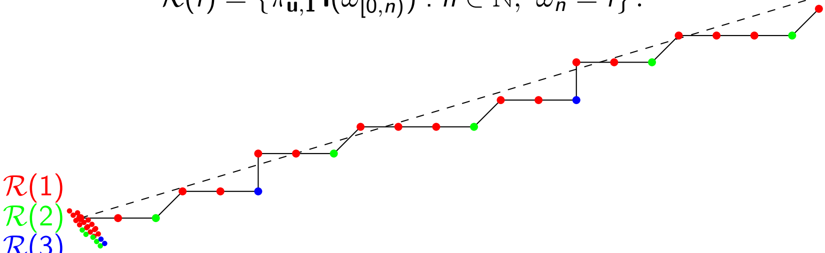
with $\mathbf{1} = (1, \ldots, 1)^t$.



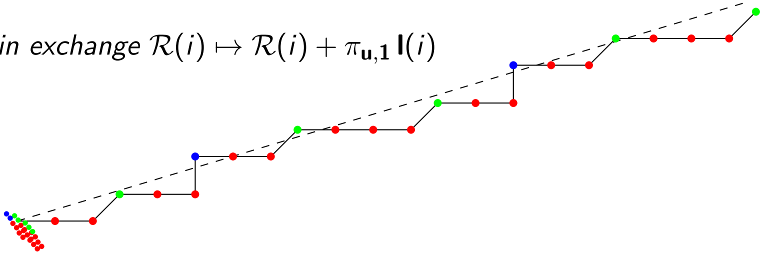
Subtiles and domain exchange

The Rauzy fractal has *subtiles* defined by

$$\mathcal{R}(i) = \overline{\{\pi_{\mathbf{u},\mathbf{1}}\, \mathbf{l}(\omega_{[0,n)}) : n \in \mathbb{N},\ \omega_n = i\}}\,.$$



domain exchange $\mathcal{R}(i) \mapsto \mathcal{R}(i) + \pi_{\mathbf{u},\mathbf{1}}\, \mathbf{l}(i)$



$\mathbf{l}(i) - \mathbf{l}(j) \in \mathcal{P}(\mathbf{1}) \cap \mathbb{Z}^d \quad \Rightarrow \quad \pi_{\mathbf{u},\mathbf{1}}\, \mathbf{l}(i) \equiv \pi_{\mathbf{u},\mathbf{1}}\, \mathbf{l}(j) \mod \mathcal{P}(\mathbf{1}) \cap \mathbb{Z}^d$

Covering and tiling of $\mathcal{P}(\mathbf{1})$

Lemma

We have

$$\left(\mathcal{P}(\mathbf{1}) \cap \mathbb{Z}^d\right) + \mathcal{R} = \mathcal{P}(\mathbf{1})$$

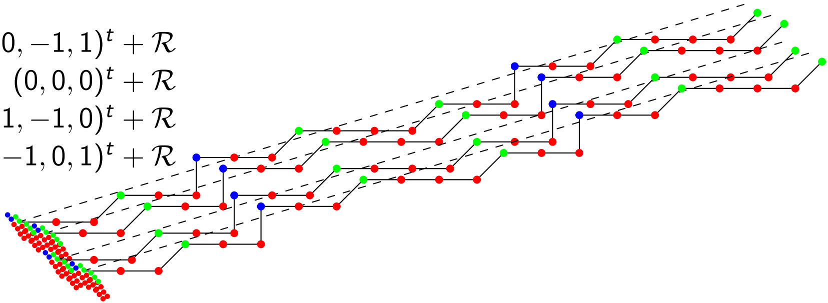
if and only if $f_1, f_2, \ldots, f_d$ *are rationally independent.*

$(0,-1,1)^t + \mathcal{R}$

$(0,0,0)^t + \mathcal{R}$

$(1,-1,0)^t + \mathcal{R}$

$(-1,0,1)^t + \mathcal{R}$



Covering and tiling of $\mathcal{P}(\mathbf{1})$

Lemma

We have

$$\left(\mathcal{P}(\mathbf{1}) \cap \mathbb{Z}^d\right) + \mathcal{R} = \mathcal{P}(\mathbf{1})$$

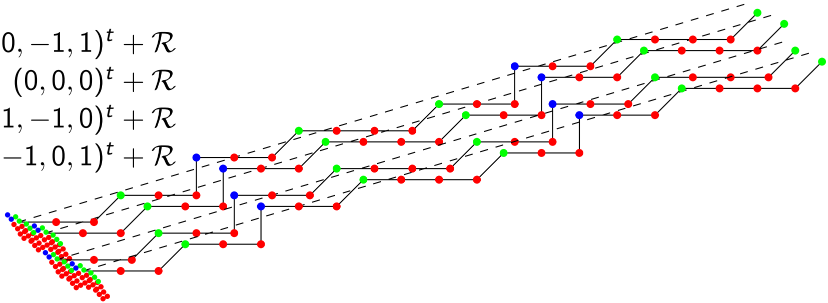
if and only if $f_1, f_2, \ldots, f_d$ are rationally independent.

$(0,-1,1)^t + \mathcal{R}$

$(0,0,0)^t + \mathcal{R}$

$(1,-1,0)^t + \mathcal{R}$

$(-1,0,1)^t + \mathcal{R}$

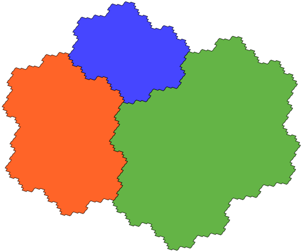


When does

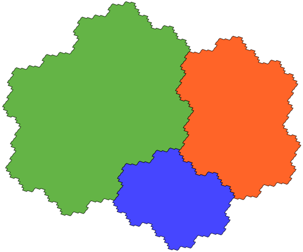
$$\mathcal{C} = \left\{\mathbf{x} + \mathcal{R} : \mathbf{x} \in \mathcal{P}(\mathbf{1}) \cap \mathbb{Z}^d\right\}$$

form a *tiling* of $\mathcal{P}(\mathbf{1})$, with $\lambda(\mathcal{R}(i) \cap \mathcal{R}(j)) = 0$ for $i \neq j$?
(λ denotes the Lebesgue measure on $\mathcal{P}(\mathbf{1})$)

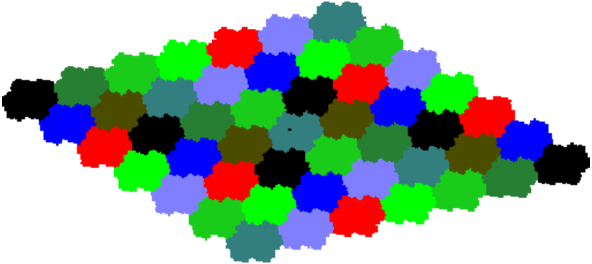
Domain exchange and tiling for the Tribonacci sequence



$\mathcal{R}(1), \mathcal{R}(2), \mathcal{R}(3)$



$\mathcal{R}(i) + \pi_{\mathbf{u},\mathbf{1}}\, \mathbf{l}(i)$



$\left\{\mathbf{x} + \mathcal{R} : \mathbf{x} \in \mathcal{P}(\mathbf{1}) \cap \mathbb{Z}^d\right\}$

Symbolic dynamical system and translation on $\mathbb{T}^{d-1}$

The *symbolic dynamical system generated by* $\omega \in A^{\mathbb{N}}$ is (X_ω, Σ), with Σ the *shift* on $A^{\mathbb{N}}$, i.e., $\Sigma((\omega_n)_{n\in\mathbb{N}}) = (\omega_{n+1})_{n\in\mathbb{N}}$, and

$$X_\omega = \overline{\{\Sigma^n(\omega) \mid n \in \mathbb{N}\}}$$

(closure w.r.t. product topology of discrete topology on A).
For most S-adic sequences, (X_ω, Σ) is uniquely ergodic.

If $\mathcal{C}$ forms a tiling of $\mathcal{P}(\mathbf{1})$, $\lambda(\mathcal{R}(i) \cap \mathcal{R}(j)) = 0$ for $i \neq j$ (and possibly other conditions), then (X_ω, Σ) is metrically conjugate to the translation by $\pi_{\mathbf{u},\mathbf{1}}\,\mathbf{l}(i)$ on $\mathcal{P}(\mathbf{1})/\big(\mathcal{P}(\mathbf{1}) \cap \mathbb{Z}^d\big) \simeq \mathbb{T}^{d-1}$, hence (X_ω, Σ) has purely discrete spectrum.

Symbolic dynamical system and translation on $\mathbb{T}^{d-1}$

The *symbolic dynamical system generated by* $\omega \in A^{\mathbb{N}}$ is (X_ω, Σ), with Σ the *shift* on $A^{\mathbb{N}}$, i.e., $\Sigma((\omega_n)_{n\in\mathbb{N}}) = (\omega_{n+1})_{n\in\mathbb{N}}$, and

$$X_\omega = \overline{\{\Sigma^n(\omega) \mid n \in \mathbb{N}\}}$$

(closure w.r.t. product topology of discrete topology on A).
For most S-adic sequences, (X_ω, Σ) is uniquely ergodic.

If $\mathcal{C}$ forms a tiling of $\mathcal{P}(\mathbf{1})$, $\lambda(\mathcal{R}(i) \cap \mathcal{R}(j)) = 0$ for $i \neq j$ (and possibly other conditions), then (X_ω, Σ) is metrically conjugate to the translation by $\pi_{\mathbf{u},\mathbf{1}}\,\mathbf{l}(i)$ on $\mathcal{P}(\mathbf{1})/\big(\mathcal{P}(\mathbf{1}) \cap \mathbb{Z}^d\big) \simeq \mathbb{T}^{d-1}$, hence (X_ω, Σ) has purely discrete spectrum.

Pisot conjecture: pure discrete spectrum in the periodic unimodular Pisot case. (Barge–Diamond '02: $d = 2$, Berthé–Jolivet–Siegel '12: periodic Arnoux-Rauzy case, Nakaishi '13, ...)

Finite balances

A pair of words $u, v \in A^*$ with $|u| = |v|$ is *C-balanced* if

$$-C \leq |u|_j - |v|_j \leq C \qquad \forall\, j \in A\,.$$

$\omega \in A^{\mathbb{N}}$ is *C-balanced* if each pair of factors u, v of ω with $|u| = |v|$ is C-balanced; ω is *finitely balanced* if it is C-balanced for some C.

Lemma

The Rauzy fractal $\mathcal{R}$ is compact if and only if ω is finitely balanced.

Finite balances

A pair of words $u, v \in A^*$ with $|u| = |v|$ is *C-balanced* if

$$-C \leq |u|_j - |v|_j \leq C \qquad \forall\, j \in A\,.$$

$\omega \in A^{\mathbb{N}}$ is *C-balanced* if each pair of factors u, v of ω with $|u| = |v|$ is C-balanced; ω is *finitely balanced* if it is C-balanced for some C.

Lemma

The Rauzy fractal $\mathcal{R}$ is compact if and only if ω is finitely balanced.

Examples (Berthé–Cassaigne–St '13)

Let ω be an Arnoux-Rauzy sequence on $A = \{1, 2, 3\}$, $\sigma_n = \tau_{t_n}$.

- If $\nexists\, m \in \mathbb{N}$ with $t_m = t_{m+1} = \cdots = t_{m+h}$ (i.e., weak partial quotients are bounded by h), then ω is $(2h+1)$-balanced.
- Let X be the set $\{1121, 1122, 12121, 12122\}$ together with the words obtained by permutations of letters.
  If $t_0 t_1 t_2 \cdots$ has no factor in X, then ω is 2-balanced.
- Cassaigne–Ferenczi–Zamboni '00 construct Arnoux-Rauzy sequences on 3 letters that are not finitely balanced, hence they are not natural codings of rotations on $\mathbb{T}^2$.

Generalised Perron-Frobenius eigenvectors

Let M_n be the incidence matrix of σ_n. An S-adic word is

- *primitive* if $\forall\, j \in \mathbb{N}\ \exists\, k > j$ such that $M_{[j,k)}$ is strictly positive,
- *S-recurrent* if the sequence $(\sigma_n)_{n\in\mathbb{N}}$ is recurrent.

If ω is primitive and S-recurrent, then by Furstenberg '60

$$\bigcap_{n\in\mathbb{N}} M_{[0,n)}\,\mathbb{R}_+^d = \mathbb{R}_+\mathbf{u} \quad \text{for some} \quad \mathbf{u} \in \mathbb{R}_+^d\,;$$

$\mathbf{u}$ is called *generalised right eigenvector*. If ω is finitely balanced, then frequencies exist and $\mathbf{u}$ is the frequency vector of ω.

Generalised Perron-Frobenius eigenvectors

Let M_n be the incidence matrix of σ_n. An S-adic word is

- *primitive* if $\forall\, j \in \mathbb{N}\ \exists\, k > j$ such that $M_{[j,k)}$ is strictly positive,
- *S-recurrent* if the sequence $(\sigma_n)_{n\in\mathbb{N}}$ is recurrent.

If ω is primitive and S-recurrent, then by Furstenberg '60

$$\bigcap_{n\in\mathbb{N}} M_{[0,n)}\,\mathbb{R}_+^d = \mathbb{R}_+\mathbf{u} \quad \text{for some} \quad \mathbf{u} \in \mathbb{R}_+^d\,;$$

$\mathbf{u}$ is called *generalised right eigenvector*. If ω is finitely balanced, then frequencies exist and $\mathbf{u}$ is the frequency vector of ω.

Let $(n_k)_{k\in\mathbb{N}}$ be an increasing subsequence of $\mathbb{N}$ with $\sigma_{[n_k,n_k+k)} = \sigma_{[0,k)}$ (which exists if ω is S-recurrent). We can choose $(n_k)_{k\in\mathbb{N}}$ such that

$$\bigcap_{n\in\mathbb{N}} \mathbb{R}_+^d\, M_{[0,n_k)} = \mathbb{R}_+\mathbf{v} \quad \text{for some} \quad \mathbf{v} \in \mathbb{R}_+^d\,;$$

$\mathbf{v}$ is called *generalised left eigenvector*.

Generalised Perron-Frobenius eigenvectors

Let M_n be the incidence matrix of σ_n. An S-adic word is

- *primitive* if $\forall\, j \in \mathbb{N}\ \exists\, k > j$ such that $M_{[j,k)}$ is strictly positive,
- *S-recurrent* if the sequence $(\sigma_n)_{n\in\mathbb{N}}$ is recurrent.

If ω is primitive and S-recurrent, then by Furstenberg '60

$$\bigcap_{n\in\mathbb{N}} M_{[0,n)}\, \mathbb{R}_+^d = \mathbb{R}_+ \mathbf{u} \quad \text{for some} \quad \mathbf{u} \in \mathbb{R}_+^d\,;$$

$\mathbf{u}$ is called *generalised right eigenvector*. If ω is finitely balanced, then frequencies exist and $\mathbf{u}$ is the frequency vector of ω.

Let $(n_k)_{k\in\mathbb{N}}$ be an increasing subsequence of $\mathbb{N}$ with $\sigma_{[n_k,n_k+k)} = \sigma_{[0,k)}$ (which exists if ω is S-recurrent). We can choose $(n_k)_{k\in\mathbb{N}}$ such that

$$\bigcap_{n\in\mathbb{N}} \mathbb{R}_+^d\, M_{[0,n_k)} = \mathbb{R}_+ \mathbf{v} \quad \text{for some} \quad \mathbf{v} \in \mathbb{R}_+^d\,;$$

$\mathbf{v}$ is called *generalised left eigenvector*. Let

$$\mathbf{u}_n = (M_{[0,n)})^{-1}\, \mathbf{u}\,, \qquad \mathbf{v}_n = \mathbf{v}\, M_{[0,n)}\,.$$

Then

$$\lim_{k\to\infty} \frac{\mathbf{u}_{n_k}}{\|\mathbf{u}_{n_k}\|} = \frac{\mathbf{u}}{\|\mathbf{u}\|}\,, \qquad \lim_{k\to\infty} \frac{\mathbf{v}_{n_k}}{\|\mathbf{v}_{n_k}\|} = \frac{\mathbf{v}}{\|\mathbf{v}\|}\,.$$

Rauzy fractals in $\mathcal{P}(\mathbf{v}_k)$ and set equations

Let
$$\omega^{(k)} = \lim_{n\to\infty} \sigma_{[k,n)}(a_n)\,, \qquad (\omega = \omega^{(0)})$$
$$\mathcal{R}_k(i) = \overline{\left\{\pi_{\mathbf{u}_k,\mathbf{v}_k}\,\mathbf{l}\big(\omega^{(k)}_{[0,n)}\big) : n \in \mathbb{N},\ \omega^{(k)}_n = i\right\}}\,.$$

Rauzy fractals in $\mathcal{P}(\mathbf{v}_k)$ and set equations

Let

$$\omega^{(k)} = \lim_{n\to\infty} \sigma_{[k,n)}(a_n)\,, \qquad (\omega = \omega^{(0)})$$

$$\mathcal{R}_k(i) = \overline{\left\{\pi_{\mathbf{u}_k,\mathbf{v}_k}\,\mathbf{l}\big(\omega^{(k)}_{[0,n)}\big) : n \in \mathbb{N},\ \omega^{(k)}_n = i\right\}}\,.$$

Then $\mathcal{R}_0(i) = \pi_{\mathbf{u},\mathbf{v}}\mathcal{R}(i)$, and we have the set equations

$$\mathcal{R}_k(i) = \bigcup_{j\in A,\, p,s\in A^*:\, \sigma_k(j)=pis} \big(\pi_{\mathbf{u}_k,\mathbf{v}_k}\mathbf{l}(p) + M_k\,\mathcal{R}_{k+1}(j)\big)\,,$$

(note that $M_k\pi_{\mathbf{u}_{k+1},\mathbf{v}_{k+1}} = \pi_{\mathbf{u}_k,\mathbf{v}_k}M_k$), i.e.,

$$M_k^{-1}\,\mathcal{R}_k(i) = \bigcup_{j\in A,\, p,s\in A^*:\, \sigma_k(j)=pis} \big(\pi_{\mathbf{u}_{k+1},\mathbf{v}_{k+1}}M_k^{-1}\,\mathbf{l}(p) + \mathcal{R}_{k+1}(j)\big)\,.$$

Rauzy fractals in $\mathcal{P}(\mathbf{v}_k)$ and set equations

Let

$$\omega^{(k)} = \lim_{n\to\infty} \sigma_{[k,n)}(a_n)\,, \qquad (\omega = \omega^{(0)})$$

$$\mathcal{R}_k(i) = \overline{\left\{\pi_{\mathbf{u}_k,\mathbf{v}_k}\,\mathbf{l}\big(\omega^{(k)}_{[0,n)}\big) : n \in \mathbb{N},\ \omega^{(k)}_n = i\right\}}\,.$$

Then $\mathcal{R}_0(i) = \pi_{\mathbf{u},\mathbf{v}}\mathcal{R}(i)$, and we have the set equations

$$\mathcal{R}_k(i) = \bigcup_{j\in A,\, p,s\in A^*:\, \sigma_k(j)=pis} \big(\pi_{\mathbf{u}_k,\mathbf{v}_k}\mathbf{l}(p) + M_k\,\mathcal{R}_{k+1}(j)\big)\,,$$

(note that $M_k\pi_{\mathbf{u}_{k+1},\mathbf{v}_{k+1}} = \pi_{\mathbf{u}_k,\mathbf{v}_k}M_k$), i.e.,

$$M_k^{-1}\,\mathcal{R}_k(i) = \bigcup_{j\in A,\, p,s\in A^*:\, \sigma_k(j)=pis} \big(\pi_{\mathbf{u}_{k+1},\mathbf{v}_{k+1}}M_k^{-1}\,\mathbf{l}(p) + \mathcal{R}_{k+1}(j)\big)\,.$$

The *discrete hyperplane* approximating $\mathcal{P}(\mathbf{v}_k)$ is

$$\Gamma(\mathbf{v}_k) = \{[\mathbf{x}, i] \in \mathbb{Z}^d \times A : 0 \le \langle \mathbf{v}_k, \mathbf{x}\rangle < \langle \mathbf{v}_k, \mathbf{l}(i)\rangle\}\,.$$

Let

$$\mathcal{C}_k = \{\pi_{\mathbf{u}_k,\mathbf{v}_k}(\mathbf{x}) + \mathcal{R}_k(i) : [\mathbf{x}, i] \in \Gamma(\mathbf{v}_k)\}\,.$$

Rauzy fractals in $\mathcal{P}(\mathbf{v}_k)$ and set equations

Let

$$\omega^{(k)} = \lim_{n\to\infty} \sigma_{[k,n)}(a_n)\,, \qquad (\omega = \omega^{(0)})$$

$$\mathcal{R}_k(i) = \overline{\left\{\pi_{\mathbf{u}_k,\mathbf{v}_k}\,\mathbf{l}\big(\omega_{[0,n)}^{(k)}\big) : n \in \mathbb{N},\ \omega_n^{(k)} = i\right\}}\,.$$

Then $\mathcal{R}_0(i) = \pi_{\mathbf{u},\mathbf{v}}\mathcal{R}(i)$, and we have the set equations

$$\mathcal{R}_k(i) = \bigcup_{j\in A,\, p,s\in A^*:\, \sigma_k(j)=pis} \left(\pi_{\mathbf{u}_k,\mathbf{v}_k}\mathbf{l}(p) + M_k\,\mathcal{R}_{k+1}(j)\right),$$

(note that $M_k\pi_{\mathbf{u}_{k+1},\mathbf{v}_{k+1}} = \pi_{\mathbf{u}_k,\mathbf{v}_k}M_k$), i.e.,

$$M_k^{-1}\,\mathcal{R}_k(i) = \bigcup_{j\in A,\, p,s\in A^*:\, \sigma_k(j)=pis} \left(\pi_{\mathbf{u}_{k+1},\mathbf{v}_{k+1}}M_k^{-1}\,\mathbf{l}(p) + \mathcal{R}_{k+1}(j)\right).$$

The *discrete hyperplane* approximating $\mathcal{P}(\mathbf{v}_k)$ is

$$\Gamma(\mathbf{v}_k) = \{[\mathbf{x}, i] \in \mathbb{Z}^d \times A : 0 \le \langle \mathbf{v}_k, \mathbf{x}\rangle < \langle \mathbf{v}_k, \mathbf{l}(i)\rangle\}\,.$$

Let

$$\mathcal{C}_k = \{\pi_{\mathbf{u}_k,\mathbf{v}_k}(\mathbf{x}) + \mathcal{R}_k(i) : [\mathbf{x}, i] \in \Gamma(\mathbf{v}_k)\}\,.$$

If M_k is unimodular, i.e., $|\det M_k| = 1$, then

$$\Gamma(\mathbf{v}_{k+1}) = \left\{[M_k^{-1}(\mathbf{x} + \mathbf{l}(p)), j] : [\mathbf{x}, i] \in \Gamma(\mathbf{v}_k),\ \sigma_k(j) = pis\right\}.$$

Contractions and convergence

Lemma

In the Arnoux-Rauzy case (and most other cases), finite balancedness implies

$$(*) \qquad \lim_{k\to\infty} \|\pi_{\mathbf{u},\mathbf{u}}(M_{[0,n_k)}\mathbf{x})\| = 0 \quad \forall\, \mathbf{x} \in \mathbb{R}^d.$$

If M_n is regular $\forall\, n \in \mathbb{N}$, $(*)$ implies that $f_1, f_2, \ldots, f_d$ are rationally independent.

Contractions and convergence

Lemma

In the Arnoux-Rauzy case (and most other cases), finite balancedness implies

$$(*) \qquad \lim_{k\to\infty} \|\pi_{\mathbf{u},\mathbf{u}}(M_{[0,n_k)}\mathbf{x})\| = 0 \quad \forall\, \mathbf{x} \in \mathbb{R}^d.$$

If M_n is regular $\forall\, n \in \mathbb{N}$, $(*)$ implies that $f_1, f_2, \ldots, f_d$ are rationally independent.

Recall $\sigma_{[n_k,n_k+k)} = \sigma_{[0,k)}$, $\lim_{k\to\infty} \frac{\mathbf{u}_{n_k}}{\|\mathbf{u}_{n_k}\|} = \frac{\mathbf{u}}{\|\mathbf{u}\|}$, $\lim_{k\to\infty} \frac{\mathbf{v}_{n_k}}{\|\mathbf{v}_{n_k}\|} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$.

Lemma

If $(*)$ holds and $\exists\, C$ such that $\omega^{(n_k)}$ is C-balanced $\forall k \in \mathbb{N}$, then

$$\lim_{k\to\infty} \mathcal{R}_{n_k}(i) = \mathcal{R}_0(i)$$

w.r.t. Hausdorff metric.

Results

Let ω be a finitely balanced, unimodular, S-recurrent, and primitive S-adic word. Assume that $\exists$ increasing subsequence $(n_k)_{k\in\mathbb{N}}$ of $\mathbb{N}$ with $\sigma_{[n_k,n_k+k)} = \sigma_{[0,k)}$ such that $\omega^{(n_k)}$ is C-balanced $\forall\, k \in \mathbb{N}$ and $\lim_{k\to\infty} \|\pi_{\mathbf{u},\mathbf{u}}(M_{[0,n_k)}\mathbf{x})\| = 0\ \forall\, \mathbf{x} \in \mathbb{R}^d$. Then

- $\mathcal{R}$ is a compact set that is the closure of its interior.
- The boundary of $\mathcal{R}$ has zero measure.
- The collections $\mathcal{C}$ and $\mathcal{C}_n$, $n \in \mathbb{N}$, form multiple tilings with same covering degree.
- (X_ω, Σ) is minimal and uniquely ergodic.

Results

Let ω be a finitely balanced, unimodular, S-recurrent, and primitive S-adic word. Assume that $\exists$ increasing subsequence $(n_k)_{k\in\mathbb{N}}$ of $\mathbb{N}$ with $\sigma_{[n_k,n_k+k)} = \sigma_{[0,k)}$ such that $\omega^{(n_k)}$ is C-balanced $\forall\, k \in \mathbb{N}$ and $\lim_{k\to\infty} \|\pi_{\mathbf{u},\mathbf{u}}(M_{[0,n_k)}\mathbf{x})\| = 0 \;\forall\, \mathbf{x} \in \mathbb{R}^d$. Then

- $\mathcal{R}$ is a compact set that is the closure of its interior.
- The boundary of $\mathcal{R}$ has zero measure.
- The collections $\mathcal{C}$ and $\mathcal{C}_n$, $n \in \mathbb{N}$, form multiple tilings with same covering degree.
- (X_ω, Σ) is minimal and uniquely ergodic.

On examples, we can show:

- The subtiles $\mathcal{R}(i)$, $i \in A$, are mutually disjoint in measure, and $\mathcal{C}$ forms a tiling of $\mathcal{P}(\mathbf{1})$, i.e., the covering degree is 1.
- (X_ω, Σ) is metrically conjugate to a translation on $\mathbb{T}^{d-1}$; in particular, its spectrum is purely discrete.

Results

Let ω be a finitely balanced, unimodular, S-recurrent, and primitive S-adic word. Assume that $\exists$ increasing subsequence $(n_k)_{k\in\mathbb{N}}$ of $\mathbb{N}$ with $\sigma_{[n_k,n_k+k)} = \sigma_{[0,k)}$ such that $\omega^{(n_k)}$ is C-balanced $\forall\, k \in \mathbb{N}$ and $\lim_{k\to\infty} \|\pi_{\mathbf{u},\mathbf{u}}(M_{[0,n_k)}\mathbf{x})\| = 0 \;\forall\, \mathbf{x} \in \mathbb{R}^d$. Then

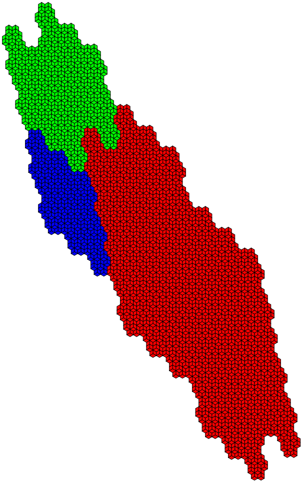
- $\mathcal{R}$ is a compact set that is the closure of its interior.
- The boundary of $\mathcal{R}$ has zero measure.
- The collections $\mathcal{C}$ and $\mathcal{C}_n$, $n \in \mathbb{N}$, form multiple tilings with same covering degree.
- (X_ω, Σ) is minimal and uniquely ergodic.

On examples, we can show:

- The subtiles $\mathcal{R}(i)$, $i \in A$, are mutually disjoint in measure, and $\mathcal{C}$ forms a tiling of $\mathcal{P}(\mathbf{1})$, i.e., the covering degree is 1.
- (X_ω, Σ) is metrically conjugate to a translation on $\mathbb{T}^{d-1}$; in particular, its spectrum is purely discrete.

Proof of tiling property: find an exclusive point of $\mathcal{C}_0$ by showing for some $i \in A$, $n \in \mathbb{N}$, that $\{[(M_{[0,n)})^{-1}\mathbf{l}(p)), j] : \sigma_{[0,n)}(j) = pis\}$ contains all elements of $\Gamma(\mathbf{v}_n)$ in a sufficiently large neighbourhood.

An example



Arnoux-Rauzy on $A = \{1,2,3\}$ with
$\sigma_n = \tau_{t_n}$, $t_0 t_1 \cdots = \lim_{k\to\infty} \varphi^k(1)$
$\varphi : 1 \mapsto 1123, 2 \mapsto 23, 3 \mapsto 123$ (Chacon)

Berthé–Cassaigne–St '13:
$\omega^{(n)}$ is 2-balanced $\forall\, n \in \mathbb{N}$

$\{[(M_{[0,13)})^{-1}\mathbf{l}(p)), j] : \sigma_{[0,13)}(j) = pis\} \; \big(= E_1^*(\sigma_{[0,13)})([\mathbf{0}, i])\big)$
for $i = 1$ (red), $i = 2$ (green), $i = 3$ (blue)