Generalised crossed products and cyclic cohomology

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Plan of the talk

Follow the title:

1. From some "twisted dynamical system" $\mathbb{Z}$,
   construct Generalised Crossed Products (GCP).
2. Introduce required (co)homologies.
3. Present our results in this setting.

Illustration: case of *Quantum Heisenberg Manifolds* (QHM).

Disclaimer: everything expressed in terms of algebras!

Outline

Dynamical systems over $\mathbb{Z}$ and algebras

No topology on algebras (for the moment)!

More top.

Geometry	Algebra
(compact) space X	functions on X, (unital) algebra $B = C(X)$
homeomorphism σ of X	automorphism σ of B

"Translation" of dynamical systems over $\mathbb{Z}$:
space X, homeomorphism σ,

$\rightsquigarrow$ *crossed product* $A := B \rtimes_\sigma \mathbb{Z}$ *i.e.*

- algebra generated by all bU for $b \in C(X)$...
- with product

$$bU \cdot b'U = b\sigma(b')U^2.$$

and $\xi \otimes \sigma(\xi') \in C(X; \mathcal{L} \otimes \sigma\mathcal{L})$.

All crossed products carry a natural action of S^1.

Example: noncommutative torus ($X = S^1$, σ: irrational rotation).

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Limit cases and examples

Generalised Crossed Products (GCP) associated to $(X, \sigma, \mathcal{L})$.

Limit cases of $A = B \rtimes_{\sigma,\mathcal{L}} \mathbb{Z}$:

- If $\mathcal{L} \to X$ is trivial: usual crossed product $A \rtimes_\sigma \mathbb{Z}$,
- If $\sigma = \mathrm{Id}_X$: get a commutative algebra, functions on S^1-principal bundle $P \to X$ generating $\mathcal{L}$.

More Details

Simple non-degenerated examples:

- $X = T^2$, $\sigma = \sigma_{\mu,\nu}$, translation by $\mu, \nu \in \mathbb{R}$.
- Line bundles $\mathcal{L} \to T^2$: classified by $c \in \mathbb{Z}$.

$\rightsquigarrow$ *Quantum Heisenberg Manifolds*: algebras $D^c_{\mu,\nu}$ indexed by

- a "topological" parameter $c \in \mathbb{Z}$,
- two "dynamical" parameters $\mu, \nu \in \mathbb{R}$.

When $\mu = \nu = 0$, recover $D^c_{0,0} = C(M^c)$, functions on the usual *Heisenberg manifolds* M^c.

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(Co)homology for algebras

Introduce topology on the algebras.

(Co)homology for algebras? Two possibilities:

Consider only topological properties $\rightsquigarrow$ K-theory.

- Use C^*-algebras as $B = C^0(X;\mathbb{C})$, continuous functions.
- $K_0(B)$ and $K_1(B)$: Abelian (discrete) groups which...
- ...classify modules (for K_0) and unitaries (for K_1) of B.
- Covariant: $B_1 \to B_2$ induces $K_*(B_1) \to K_*(B_2)$.

Analog of De Rham homology $\rightsquigarrow$ Cyclic cohomology.

- Requires a differential structure on X...
- ...translates into a Fréchet structure for $\mathscr{B} = C^\infty(X;\mathbb{C})$.
- HP^0 and HP^1: vector spaces of multilinear forms on $\mathscr{B}$.
- Contravariant: $\mathscr{B}_1 \to \mathscr{B}_2$ induces $HP^*(\mathscr{B}_1) \leftarrow HP^*(\mathscr{B}_2)$.

For $\mathscr{B} = C^\infty(X;\mathbb{C})$, recover:

$$HP^0(\mathscr{B}) = H^{\mathrm{dR}}_{\mathrm{even}}(X) \qquad HP^1(\mathscr{B}) = H^{\mathrm{dR}}_{\mathrm{odd}}(X)$$

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Example: quantum Heisenberg manifolds

Two different versions of quantum Heisenberg manifolds:

- C^*-algebras $D^c_{\mu,\nu}$ properly speaking. K-theory:

Proposition (Abadie, 1995)

$$K_0(D) = \mathbb{Z}^3 \oplus \mathbb{Z}/c\mathbb{Z} \qquad K_1(D) = \mathbb{Z}^3.$$

- "Smooth" algebras $\mathscr{D}^c_{\mu,\nu}$. Cyclic cohomology:

Proposition (G., 2011)

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Duality and Chern-Connes pairings

Combining both: Chern-Connes pairings. For good $\mathscr{B} \subseteq B$,

$$\text{for } K \in K_j(B) \text{ and } \varphi \in HP^j(\mathscr{B}), \qquad \langle K, \varphi \rangle \in \mathbb{C}$$

Exists for $j = 0, 1$. "Bilinear" in φ and K. Manageable.

- Improves knowledge of algebras. For QHM, recover μ, ν:

Theorem (Abadie, 2000 – G., 2011)

Bilinear pairings $K_*(D)/HP^*(\mathscr{D})$, even and odd cases

	τ	$\varphi_{1,3}$	$\varphi_{2,3}$
$[P_0]$	1	0	0
$[P_1]$	2μ	1	0
$[P_2]$	2ν	0	1

	φ_1	φ_2	$\varphi_{1,2,3}$
$[U_1]$	1	0	0
$[U_2]$	0	1	0
$[U_3]$	2ν	-2μ	1

About the proof

Particular cases:

- Evaluation of trace on K-theory (Bellissard's gap label).
- Kubo formula for Quantum Hall Effect (QHE).

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Exact sequences for K_* and HP^* – an overview

If $0 \to A \to B \to C \to 0$ is an exact sequence of C^*-algebras, the following is exact:

$$\begin{array}{ccccc} K_0(A) & \longrightarrow & K_0(B) & \longrightarrow & K_0(C) \\ \uparrow & & & & \downarrow \\ K_1(C) & \longleftarrow & K_1(B) & \longleftarrow & K_1(A). \end{array}$$

Theorem (Cuntz & Quillen, 1994)

If $0 \to \mathscr{A} \to \mathscr{B} \to \mathscr{C} \to 0$ exact (algebras with sections), the following is exact:

$$\begin{array}{ccccc} HP^0(\mathscr{A}) & \longleftarrow & HP^0(\mathscr{B}) & \longleftarrow & HP^0(\mathscr{C}) \\ \downarrow & & & & \uparrow \\ HP^1(\mathscr{C}) & \longrightarrow & HP^1(\mathscr{B}) & \longrightarrow & HP^1(\mathscr{A}). \end{array}$$

K-theory for Generalised Crossed Products

For a C^*-algebraic GCP, Pimsner-Voiculescu type exact sequence:

Theorem (Pimsner, 1997 – Abadie, Eilers & Exel, 1998)

$$\begin{array}{ccccc} K_0(B) & \longrightarrow & K_0(B) & \longrightarrow & K_0(B \rtimes_{\sigma,\mathcal{L}} \mathbb{Z}) \\ \uparrow \partial & & & & \downarrow \partial \\ K_1(B \rtimes_{\sigma,\mathcal{L}} \mathbb{Z}) & \longleftarrow & K_1(B) & \longleftarrow & K_1(B). \end{array}$$

The maps $K_j(B) \to K_j(B)$ are defined out of $\mathcal{L}$ and σ.

We want a similar statement for HP^*, *i.e.*

1. a "smooth version" $\mathscr{B}\hat{\rtimes}\mathbb{Z}$ of $B \rtimes_{\sigma,\mathcal{L}} \mathbb{Z}$;
2. a 6-term exact sequence for HP^*;
3. a "compatibility" between these two 6-term exact sequences.

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Main results: hypotheses

If $B \rtimes_{\sigma,\mathcal{L}} \mathbb{Z}$ is a GCP s.t. X is compact and G Lie group acting on the GCP, then

- smooth version $\mathscr{B}\hat{\rtimes}\mathbb{Z}$ of $B \rtimes_{\sigma,\mathcal{L}} \mathbb{Z}$ can be defined;
- there is a linearly split exact sequence

$$0 \to \mathscr{C} \to \mathscr{T} \to \mathscr{B}\hat{\rtimes}\mathbb{Z} \to 0. \qquad \text{(Toeplitz Extension)}$$

Furthermore, if $\mathscr{B}\hat{\rtimes}\mathbb{Z}$ is tame, ▸ Definition

- then the following holds:

Proposition: equalities of HP^* (G. & Grensing, 2011)

Under the same hypotheses,

$$HP^*(\mathscr{C}) = HP^*(\mathscr{B}) \qquad HP^*(\mathscr{T}) = HP^*(\mathscr{B}).$$

$\rightsquigarrow$ Holds for more general functors (Cuntz' *kk*-theory). ▸ Details

Main results: statement

Theorem (G. & Grensing, 2011)

Under the previous hypotheses (in part. $\mathscr{B}\hat{\rtimes}\mathbb{Z}$ tame),

1. Exact sequence:

$$\begin{array}{ccccc} HP^0(\mathscr{B}) & \longleftarrow & HP^0(\mathscr{B}) & \longleftarrow & HP^0(\mathscr{B}\hat{\rtimes}\mathbb{Z}) \\ \downarrow \# & & & & \uparrow \# \\ HP^1(\mathscr{B}\hat{\rtimes}\mathbb{Z}) & \longrightarrow & HP^1(\mathscr{B}) & \longrightarrow & HP^1(\mathscr{B}). \end{array}$$

2. Transfer formula: $\forall K \in K_i(B \rtimes_{\sigma,\mathcal{L}} \mathbb{Z})$, $\forall \varphi \in HP^{i+1}(\mathscr{B})$,

$$\langle [K], \#\varphi \rangle = 2i\pi \langle \partial[K], \varphi \rangle.$$

- Lie group action: only required to define $\mathscr{B}\hat{\rtimes}\mathbb{Z}$.
- Transfer formula: consequence of (Toeplitz Extension) and results by Nistor (1997). Extends Nest (1988).

Case of QHM

Tameness – statement

Hermitian line bundle $\mathcal{L} \to X$, *i.e.* pointwise scalar product

$$_B\langle \cdot, \cdot \rangle \colon C^\infty(X;\mathcal{L}) \times C^\infty(X;\mathcal{L}) \to C^\infty(X).$$

Definition (frame of $\Lambda \to X$)

A finite family $v_1, \ldots, v_N$ in $C^\infty(X;\Lambda)$ such that

$$\sum_\omega {}_B\langle v_\omega, v_\omega \rangle = 1.$$

Denote $\mathscr{E}^m := C^\infty(X; \mathcal{L} \otimes \sigma\mathcal{L} \otimes \cdots \otimes \sigma^{m-1}\mathcal{L})$, for $m \in \mathbb{N}$.

The smooth algebra is equipped with continuous seminorms p.

Definition (Tame GCP)

- For all $m \in \mathbb{N}$, $\mathscr{E}^m$ admit a frame $(v_\omega^m)_\omega$;
- the length of the frames (v_ω^m) is *uniformly* bounded;
- For all seminorm p on $\mathscr{B} \hat{\rtimes} \mathbb{Z}$, the sequence $p(v_\omega)$ has polynomial growth.

Tameness – sufficient condition and examples

Consider a GCP with X compact, $B = C(X)$, Hermitian line bundle $\mathcal{L} \to X$ and $\sigma \colon B \to B$ s.t.

◂ Back

- a finite dim. Lie group G acts on B...
- ...and on E, yielding an action $\alpha \colon G \curvearrowright A = B \rtimes_{\sigma,\mathcal{L}} \mathbb{Z}$;
- assume $\sigma = \alpha_{g^{-1}}$ for some $g \in G$ and
- $\forall k \in \mathbb{N}$, $\| \mathrm{Ad}_{g^k} \| \leqslant Ck^d$ (adjoint action $\mathrm{Ad} \colon G \curvearrowright \mathfrak{g}$),

then

Proposition (G.)

1. the G-smooth elements $\mathscr{A}$ of A form a smooth GCP,
2. $\mathscr{A}$ is tame for an explicit family $\left(v_\omega^{(l)}\right)_{\omega,l}$ of frames.

- The index ω correspond to a (finite) trivialisation of $\mathcal{L}$.
- The previous properties are satisfied by QHM.
- Growth condition met by compact Lie groups G.

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QHM as tame smooth GCP

A general construction of "smooth algebras": if

- G is a Lie group, A is a C^*-algebra...
- ...and $\alpha\colon G \curvearrowright A$, pointwise continuous action

then

$$\mathscr{A} := \left\{ a \in A : g \longmapsto \alpha_g(a) \text{ is } C^\infty \right\} \subseteq A,$$

is a dense subalgebra of A and $K_j(\mathscr{A}) = K_j(A)$ ("good" inclusion).

For QHM, the *Heisenberg group* H_1 acts by α on $D^c_{\mu,\nu}$, thus defining $\mathscr{D}^c_{\mu,\nu}$ as above. Moreover,

- the defining translation Reminder $\sigma = \tau_{\mu,\nu}$ on $B = C(T^2)$ of $D^c_{\mu,\nu}$ extends to α_g acting on all of $D^c_{\mu,\nu}$...
- ... for some $g = (\mu, \nu, t) \in H_1$ satisfying the growth condition $\|\operatorname{Ad}_{g^k}\| \leqslant Ck^d$ for all $k \in \mathbb{N}$.

Hence, our main theorem Reminder applies to QHM,

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For more details...

- Exact sequence in HP^*:

O. G. and M. Grensing
Six-Term Exact Sequences for Smooth Generalized Crossed Products
to appear in J. Noncommut. Geom.
http://arxiv.org/abs/1111.2154

- Particular case of QHM:

O. G.
Pairings, K-theory and Cyclic Cohomology for Quantum Heisenberg Manifolds
to appear in J. Noncommut. Geom.
http://arxiv.org/abs/1011.6287

. . .

Result for General Functors

Denote by $\mathcal{K}$ the smooth compact operators.

Theorem (Grensing, G. – 2011)

If

- H is a split-exact, diffeotopy-invariant and $\mathcal{K}$-stable functor from locally convex algebras to abelian groups,
- $\mathscr{B}\hat{\rtimes}\mathbb{Z}$ is a tame smooth GCP,

then

$$H(\mathscr{C}) \simeq H(\mathscr{A}) \qquad\qquad H(\mathscr{T}) \simeq H(\mathscr{A}).$$

Proof: use Cuntz' techniques for *kk*-theory, *e.g.* Morita contexts. Steps similar to Pimsner's proof for *KK*-theory.

1. Morita bicontext between $\mathscr{A}$ and $\mathscr{C}$.
2. Quasihomomorphism required for $H(\mathscr{T}) \simeq H(\mathscr{A})$.

Back

Generalised crossed products: general definition

Take γ a *pointwise continuous* action of $S^1 = \mathbb{R}/\mathbb{Z}$ on A.

$\gamma\colon S^1 \curvearrowright A$, *gauge action* and C^*-algebra A, *gauged algebra*.

- For any $a \in A$, $t \mapsto \gamma_t(a)$ 1-periodic, Banach-valued continuous function.
- Fourier series: introduce the subspaces A_n, $n \in \mathbb{Z}$

$$A_n = \left\{ a \in A \middle| \forall t \in \mathbb{R}, \gamma_t(a) = e^{i2\pi nt} a \right\} \dots$$

- "$\cdots \oplus A_{-2} \oplus A_{-1} \oplus A_0 \oplus A_1 \oplus A_2 \oplus \cdots$" is dense in A.

Definition

A is a GCP if it is generated (as C^*-algebra) by $B := A_0$ and A_1.

Back

First Definition

Previous properties:

$$*(A_n) = A_{-n} \qquad\qquad A_n A_m \subseteq A_{n+m}$$

Consequences:

- A_0 is a sub-C^*-algebra of A;
- A_n are bimodules over A_0.

In fact, A_0-valued scalar products on A_n:

$$\langle \xi, \eta \rangle_{A_0} = \xi^* \eta \qquad\qquad {}_{A_0}\langle \xi, \eta \rangle = \xi \eta^*.$$

The A_n are *Hilbert bimodules* over A_0. ▸ Definition

Set $B := A_0$ (*basis algebra*) and $E := A_1$,

Definition (: Generalised Crossed Product – I)

A gauged C^*-algebra A s.t. B and E generate A is called *generalised crossed product* and denoted by $B \rtimes_{\sigma,\mathcal{L}} \mathbb{Z}$.

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Commutative Case: Steps of the Proof

Steps of the proof:

- Serre-Swan theorem: $E \leftrightsquigarrow$ vector bundle $\mathcal{L} \to X$.
- Character φ of $B \rtimes_{\sigma,\mathcal{L}} \mathbb{Z} \rightsquigarrow$ character $b \mapsto b(x_0)$ of B.
  This is the map $P \to X$.
- $\mathrm{ev}_{x_0} : B \rtimes_{\sigma,\mathcal{L}} \mathbb{Z} \to \mathbb{C} \rtimes_{\mathcal{L}_{x_0}} \mathbb{Z}$ s.t. $\varphi = \psi \circ \mathrm{ev}_{x_0}$.
  Since $B \rtimes_{\sigma,\mathcal{L}} \mathbb{Z}$ is commutative, we must have $\dim \mathcal{L} = 1$.
- Find section ξ_1 of $\mathcal{L}$ with $\langle \xi_1, \xi_1 \rangle = 1$, *locally*.
  Consequently, $\varphi(\xi_1)^* \varphi(\xi_1) = \varphi(\langle \xi_1, \xi_1 \rangle) = 1$
  which shows that $\varphi(\xi_1) \in U(1)$.
- Write any $a \in B \rtimes_{\sigma,\mathcal{L}} \mathbb{Z}$ as $\sum b_n \, (\xi_1)^n$, *locally*.
- For any character φ there is $\lambda \in U(1)$:

$$\varphi\left(\sum b_n \, (\xi_1)^n\right) = \sum b_n(x_0) \lambda^n.$$

- Action of γ_t: $\varphi(\gamma_t(b)) = b$ and $\varphi(\gamma_t(\xi_1)) = e^{i2\pi t}\lambda$.

Back