Pisot Conjecure and Rauzy fractals

Kentaro Nakaishi

Janurary 14, 2013 at CIRM

Substitution

We may set $\mathcal{A} = \{1, 2, \cdots, d\}$.
Denote by $\mathcal{A}^*$ the set of all finite words over $\mathcal{A}$.

A substitution (over $\mathcal{A}$) is a mapping $\sigma : \mathcal{A} \to \mathcal{A}^*$.

- σ extends to $\mathcal{A}^*$ by requiring $\sigma(w_1 w_2) = \sigma(w_1)\sigma(w_2)$.
- A substitution σ is primitive if $\exists n > 0$ so that for $\forall i, j \in \mathcal{A}$ the sequence $\sigma^n(i)$ contains j.
- A substitution σ naturally induces a mapping on $\mathcal{A}^{\mathbb{N}}$:

$$\sigma(u) := \sigma(u_0)\sigma(u_1)\cdots \quad \text{if } u = (u_k)_{k\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}}.$$

By abuse of language, we use the same symbol σ even when we let σ act on $\mathcal{A}^{\mathbb{N}}$.

Substitution dynamical system

Suppose that a primitive substitution σ has a fixed point u:$\sigma(u) = u$.

Example (Morse-Tue substitution)

For $\sigma(1) = 12, \sigma(2) = 21$,

$$u = \lim_{n\to\infty} \sigma^n(1) = 1221211221121221\cdots$$

is a fixed point.

WE WANT TO STUDY RECURRENCE PROPERTIES OF u.

$$\overline{\mathcal{O}(u)} = \overline{\{S^n u : \forall n \geq 0\}} \subset \mathcal{A}^{\mathbb{N}}.$$

The subshift $(\overline{\mathcal{O}(u)}, S)$ is called the substitution dynamical system for σ.

- $(\overline{\mathcal{O}(u)}, S)$ is minimal.
- $\exists \nu$ a unique S-invariant ergodic measure on $\overline{\mathcal{O}(u)}$.

Spectrum

For a measure-preserving system $(\overline{O(u)}, S, \nu)$, define the unitary operator $U_S : L^2(\nu) \to L^2(\nu)$ by

$$(U_S f)(x) := f(Sx).$$

- A complex number λ is an eigenvalue of S if $\exists f \in L^2(\nu)$ non-zero such that $U_S f = \lambda f$. This f is an eigenfunction of S corresponding to λ.
- We say that the substitution dynamical system $(\overline{O(u)}, S, \nu)$ has discrete spectrum if there exist eigenfunctions of S which forms an orthonormal basis for $L^2(\nu)$.

Theorem (Halmos)

If an ergodic measure-preserving system has discrete spectrum, then it is isomorphic to some ergodic rotation on a compact abelian group.

Variation of spectral type

Unlike hyperbolic dynamical systems,

Theorem (1978 Dekking-Keane)

The substitution dynamical system for a primitive substitution is never strongly mixing.

The spectral type varies from the weakly mixing one to rotation, depending on the substitution.

Geometrisation of substitution sequences

Let $E_0(\sigma) : \mathbb{R}^d \to \mathbb{R}^d$ be a linear mapping, sometimes called *abelianization*, defined by

$$E_0(\sigma) = [f(\sigma(1)), f(\sigma(2)), \cdots, f(\sigma(d))]$$

where $f : \mathcal{A}^* \to \mathbb{Z}^d$ is a homomorphism defined by

$$f(w_1 w_2 \cdots w_k) = \mathbf{e}_{w_1} + \mathbf{e}_{w_2} + \cdots + \mathbf{e}_{w_k}$$

with $f \circ \sigma = E_0(\sigma) \circ f$.
Primitivity is equivalent to the fact that $E_0(\sigma)^n$ is positive.

Pisot substitution

An algebraic integer $\lambda > 1$ is a Pisot(-Vijayaraghavan) number if the other roots are less than one in modulus.

Example

$\lambda^2 - \lambda - 1 = 0$(golden mean), $\lambda^3 - \lambda^2 - \lambda - 1 = 0$(tribonacci),etc.

A primitive substitution σ is an irreducible Pisot substitution if the characteristic polynomial of $E_0(\sigma)$ is irreducible and its maximal eigenvalue is a Pisot number.

To study the spectrum, we may assume that any irreducible Pisot substitution has a fixed point.

Pisot Conjecture

If σ is a (unimodlar), irreducible Pisot substitution, then the subsitution dynamical system for σ has discrete spectrum.

–Classification problem of recurrent sequences in ergodic theory
–Solved for $\#\mathcal{A} = d = 2$ (2003 Hollander-Solomyak+Barge-Diamond)
–Solved for special cases of $d \geq 3$

Geometric representation of the substitution dynamical system

- $(d = 2)$ "cut & projection method"(folklore).
  $\exists$ 2-interval exchange transformation isomorphic to the substitution dynamical system for an irreducible Pisot substitution σ.
- $(d = 3)$1982 Rauzy's breakthrough
  $\sigma_R(1) = 12, \sigma_R(2) = 13, \sigma_R(3) = 1$(Rauzy substitution)
- $(d \geq 3)$2001 Ito-Arnoux generalized Rauzy's framework into unimodular, irreducible Pisot substitutions with a general combinatorial condition.

Rauzy-Ito-Arnoux framework

Let $\mathcal{P}$ be the orthogonal subspace to a positive eigenvector of ${}^tE_0(\sigma)$: $E_0(\sigma)\mathcal{P} = \mathcal{P}$.
Let $\pi : \mathcal{P} \to \mathbb{R}^d$ be the projection along the maximal eigenvector for $E_0(\sigma)$.
Denote the fixed point of σ by $u = u_0 u_1 \cdots u_k \cdots$.

Define Rauzy fractal X (for σ) by

$$X = \overline{\{\pi f(u_0 u_1 \cdots u_k) |\, k = 0, 1, 2, \cdots\}} = \overline{\{\pi \sum_{j=0}^{k} \mathbf{e}_{u_j} |\, k = 0, 1, 2, \cdots\}}$$

Set also

$$X_i' = \overline{\{\pi f(u_0 u_1 \cdots u_k) |\, u_k = i \text{ for some k}\}}$$

and

$$X_i = \overline{\{\pi f(u_0 u_1 \cdots u_{k-1}) |\, u_k = i \text{ for some k}\}}$$

for $1 \le i \le d$.

Then

$$X = \bigcup_{i \in \mathcal{A}} X'_i = \bigcup_{i \in \mathcal{A}} X_i.$$

These two partitions are related to each other by a translation

$$X'_i - \pi(\mathbf{e}_i) = X_i \quad \text{for } 1 \leq i \leq d.$$

Define the domain exchange transformation $T : X \to X$ by

$$T : X'_i \to X_i, \; T(x) = x - \pi(\mathbf{e}_i)$$

and its inverse transformation by

$$T^{-1} : X_i \to X'_i, \; T^{-1}(x) = x + \pi(\mathbf{e}_i).$$

Let Γ_0 be the discrete subgroup generated by translations

$$\pi(\mathbf{e}_i - \mathbf{e}_j) \quad (1 \leq i < j \leq d)$$

acting on $\mathcal{P}$.

Every discrete group of isometries of a metric space is properly discontinuous. ($\Rightarrow$ fundamental domain)

Denote the canonical projection by $p : \mathcal{P} \to \mathcal{P}/\Gamma_0$ and the quotient space $\mathcal{P}/\Gamma_0$ by $\mathbb{T}^{d-1}$.

Rauzy's case

$$\begin{array}{ccc} \overline{O(u)} & \xrightarrow{S} & \overline{O(u)} \\ \downarrow & & \downarrow \\ X & \xrightarrow{T} & X \\ iso.\downarrow & & \downarrow iso. \\ \mathbb{T}^2 & \xrightarrow{\tau} & \mathbb{T}^2 \end{array}$$

- Lebesgue measure on X is T-invariant.
- $(\overline{O(u)}, S, \nu)$ is isomorphic to (X, T, m) where m is the normalised Lebesuge measure on X.
- (T, X) singly covers a minimal translation on 2-torus (Rauzy fractal X is the closure of a ”fractal” fundamental domain).

Any ergodic rotation w.r.t Haar measure has discrete spectrum. $\Rightarrow$ Pisot conjecture holds for this case.

When we pay attention to a particular letter $w_k^{(j)}$ in $\sigma^n(j)$, we write $\sigma^n(j) = P_{k,n}^{(j)} w_k^{(j)} S_{k,n}^{(j)}$, where $P_{k,n}^{(j)}$ and $S_{k,n}^{(j)}$ represent its prefix and suffix respectively.

Definition (Strong coincidence conditon)

A substitution σ has a (*positive*) *strong coincidence of order* 1 for $i, j \in \mathcal{A}$ if there exists $k > 0$ such that $w_k^{(i)} = w_k^{(j)}$ and $f(P_{k,1}^{(i)}) = f(P_{k,1}^{(j)})$. It has a (*positive*) *strong coincidence* for $i, j \in \mathcal{A}$ if, for some $n > 0$, σ^n has a strong coincidence of order 1 for $i, j \in \mathcal{A}$. A substitution σ has a (*positive*) *strong coincidence for all letters* if it has a strong coincidence for any pair $i, j \in \mathcal{A}$.

Theorem (2001 Arnoux-Ito)

Let σ be an irreducible unimodular Pisot substitution over $\mathcal{A}$. If σ has a strong coincidence, then there exists a dynamical system (X, T, ν), the domain exchange system, so that it is measure-theoretically isomorphic to $(\overline{O(u)}, S, \nu)$ and is semi-conjugate to a minimal traslation on the torus $(\mathbb{T}^{d-1}, \tau)$. Moreover (X, T) is a finite extension of $(\mathbb{T}^{d-1}, \tau)$.

$$\begin{array}{ccc}
\overline{O(u)} & \xrightarrow{\quad S \quad} & \overline{O(u)} \\
\phi^{-1}\big\downarrow & & \big\downarrow\phi^{-1} \\
X & \xrightarrow{\quad T \quad} & X \\
p\big\downarrow & & \big\downarrow p \\
\mathbb{T}^{d-1} & \xrightarrow{\quad \tau \quad} & \mathbb{T}^{d-1}
\end{array}$$

Theorem (N)

Pisot conjecture is true for any irreducible, unimodular Pisot substitution with strong coincidence condition.

- The domain exchange transformations generate the discrete group. But the converse is NOT true!

$$T(x) = x - \pi(\mathbf{e}_i), \quad \Gamma_0 = \left\{ x + \sum c_{ij}\pi(\mathbf{e}_i - \mathbf{e}_j) : c_{ij} \in \mathbb{Z} \right\}.$$

- It may not be enough to take the quotient by Γ_0 to see the structure of the substitution dynamical system in Rauzy fractals.

Thus, we construct a discrete flow isomorphic to the domain exchange system instead of the toral translation.

$$\begin{array}{ccc}
\overline{O(u)} & \xrightarrow{\ S\ } & \overline{O(u)} \\
\phi^{-1}\downarrow & & \downarrow\phi^{-1} \\
X & \xrightarrow{\ T\ } & X \\
iso.\downarrow & & \downarrow iso. \\
\tilde{X} & \xrightarrow{\ \tilde{T}\ } & \tilde{X}
\end{array}$$

Pseudo-distance on Rauzy fractals

Remember that a function $d_* : X \times X \to \mathbb{R}$ is called a pseude-distance if it satisfies that for $\forall x, y, z \in X$

$$(1)\ d_*(x,y) \geq 0,\ \ (2)\ x = y \Rightarrow d_*(x,y) = 0,$$

$$(3)\ d_*(x,y) = d_*(y,x),\ \ (4)\ d_*(x,y) \leq d_*(x,z) + d_*(z,y).$$

Notice that if you replace "$\Rightarrow$" by "$\Leftrightarrow$"in (2), then d_* is called a distance on X.

If you have a pseudo-distance d_* on X, you can make a metric space from X(General topology):

Define an equivalent relation on X by

$$x \sim y \iff d_*(x, y) = 0.$$

Denote the quotient space by $\tilde{X} = X\backslash \sim$. Then you can define the distance d on $\tilde{X}$ by

$$d([x], [y]) := d_*(x, y) \quad \forall [x], [y] \in \tilde{X}.$$

Example

$\mathbb{T}^2 = [0,1]^2 / \sim$

How to introduce the distance on 2-torus from a pseudo-distance:

$$d^*(x,y) = \min\{|x - \gamma(y)| : \gamma \in \Gamma_0\} \quad (x, y \in [0,1]^2).$$

where $\Gamma_0 =< \gamma_1, \gamma_2 >$ and

$$\gamma_1(a,b) = (a+1,b), \quad \gamma_2(a,b) = (a,b+1).$$

$\Rightarrow$

$$d^*(x,y) = 0 \Leftrightarrow x \sim y \text{ (identification on the boundary)}$$

FOR THE MOMENT, SUPPOSE THAT WE HAVE SUCH A PSEUDO-DISTANCE d_* ON RAUZY FRACTAL X THAT

(P1) (isometry) $d_*(Tx, Ty) = d_*(x, y)$,

(P2) $d_*(x, y) = 0 \Leftrightarrow x = y$ or $x, y \in \partial X$ with $x - y \in \Gamma_0$.

(P3) The quotient $\tilde{X}$ is compact and
$\tilde{T} : \tilde{X} \to \tilde{X}$ by $\tilde{T}[x] := [Tx]$ is minimal.

- $(\tilde{X}, d)$ is a compact metric space.
- Define $\tilde{U} : \tilde{X} \to \tilde{X}$ by $\tilde{U}[x] := [T^{-1}x]$. It is easy to see that $\tilde{T}\tilde{U} = \tilde{U}\tilde{T} = id_{\tilde{X}}$. Thus $\tilde{T}$ is invertible.
- (P1) implies that $\tilde{T}$ is an isometry: $d(\tilde{T}[x], \tilde{T}[y]) = d([x], [y])$.
  $\Rightarrow (\tilde{X}, \tilde{T})$ is a homeomorphism on a compact metric space.

We call $(\tilde{X}, \tilde{T})$ the domain exchange flow to emphasize its invertibility.

Theorem (Halmos-von Neumann)

Let $\tilde{T} : \tilde{X} \to \tilde{X}$ be a homeomorphism on a compact metric space. The followings are equivalent.

(i) *$\tilde{T}$ is topologically transitive($\exists [x_0] \in \tilde{X}$ $\mathcal{O}([x_0]) = \{\tilde{T}^n[x_0]\}_{n\in\mathbb{Z}}$ is dense in $\tilde{X}$) and is an isometry for some metric d on $\tilde{X}$*

(ii) *$\tilde{T}$ is topologically conjugate to a minimal rotation on a compact abelian metric group.*

A group structure on $\tilde{X}$ can be introduced through $\mathcal{O}([x_0])$:
a multiplication $*$ is defined by $\tilde{T}^n[x_0] * \tilde{T}^m[x_0] := \tilde{T}^{n+m}[x_0]$. The isometry $d(\tilde{T}[x], \tilde{T}[y]) = d([x], [y])$ allows group operations to extend on $\tilde{X}$. For any $[x] \in \tilde{X}$,
$\tilde{T}[x] = \lim_{n\to\infty} \tilde{T}\tilde{T}^n[x_0] = \lim_{n\to\infty} \tilde{T}[x_0] * \tilde{T}^n[x_0] = (\tilde{T}[x_0]) * [x]$,
thus a rotation.

(P1),(P2) and (P3) guarantees that $\tilde{T}$ satisfies (i) of Halmos-von Neumann theorem. So $\tilde{T}$ is conjugate to a minimal rotation on a compact abelian group.

Let G be a compact topological group. Then there exists a unique probability measure m on G so that

$$m(xE) = m(E)\ \forall x \in G\ \forall E \in \mathcal{B}(G).$$

This measure is called the normalisesd Haar measure.
The T- invariant ergodic measure ν on X induces a $\tilde{T}$-invariant measure μ on $\tilde{X}$: $\mu = \nu \circ q^{-1}$ where $q : X \to \tilde{X}$, $q(x) = [x]$.
We can prove

Lemma

The induced invariant measure $\mu = \nu \circ q^{-1}$ is the normalised Haar measure on $\tilde{X}$.

(P2) implies that the difference between $\tilde{X}$ and X occurs only on the boundary ∂X and the interior $\mathrm{int} X$ is preserved by q.
Since $\nu(\partial X) = 0$, we conclude

Lemma

$(\tilde{X}, \tilde{T}, \mu)$ is measure-theoretically isomorphic to (X, T, ν).

$$\begin{array}{ccc} (X,\nu) & \xrightarrow{T} & (X,\nu) \\ q \downarrow & & \downarrow q \\ (\tilde{X},\mu) & \xrightarrow{\tilde{T}} & (\tilde{X},\mu) \end{array}$$

We refer to the following two standard theorems in Ergodic theory.

Theorem

Let G be a compact group and $\tilde{T}(g) = ag$ be a rotation of G. Then $\tilde{T}$ is ergodic w.r.t the normalised Haar measure iff $\{a^n\}_{n\in\mathbb{Z}}$ is dense in G.

Theorem

Let $\tilde{T}(g) = ag$ be an ergodic rotation of a compact abelian group G. Then $\tilde{T}$ has discrete spectrum

Consequently, $(\tilde{X}, \tilde{T}, \mu)$ has discrete spectrum and so do (X, T, ν) and $(\overline{\mathcal{O}(u)}, S, \nu)$.
Which proves Pisot conjecture for irreducible, unimodular Pisot substitutions with strong coincidence.

How to construct such a nice pseudo-distance

WE HOPE TO CONSTRUCT A SIMILAR PSEUDO-DISTANCE ON RAUZY FRACTALS THAT IDENTIFIES THE BOUNDARY JUST AS WE SEE IN THE CASE OF 2-TORUS.

Difficulties:

- How to identify the fractal bounday?
- No guarantee that X is the closure of a fundamental domain.

Theorem (Set equation)

Given $i \in \mathcal{A}$ and $n \geq 1$,
the following set equation holds:

$$X'_i = \sum_{j \in \mathcal{A}} \sum_{\sigma^n(j) = P^{(j)}_{k,n} i S^{(j)}_{k,n}} E_0(\sigma)^n X'_j - \pi(f(S^{(j)}_{k,n})).$$

Similarly

$$X_i = \sum_{j \in \mathcal{A}} \sum_{\sigma^n(j) = P^{(j)}_{k,n} i S^{(j)}_{k,n}} E_0(\sigma)^n X_j + \pi(f(P^{(j)}_{k,n})).$$

From the set equation, $E_0(\sigma)^{-n}X$ makes a local multi-tiling $\mathcal{T}$ on $\mathcal{P}$ for each $n \geq 0$:

$$\mathcal{T} = \{X_j + \pi(f(P^{(i)}_{k,n})) : \forall j \in \mathcal{A}\ \forall i \in \mathcal{A}\ \forall k \geq 1\ \sigma^n(i) = P^{(i)}_{k,n} j S^{(i)}_{k,n}\}.$$

Theorem (Arnoux-Ito,Feng-Furukado-Ito-Wu)

Under the same assumption of [Arnoux-Ito], $\mathcal{T}$ is a local tiling. In other words, $|X_i + \pi(\mathbf{p}) \cap X_j + \pi(\mathbf{q})| = 0$ for any distinct pair $X_i + \pi(\mathbf{p}), X_j + \pi(\mathbf{q}) \in \mathcal{T}$. Furthermore, $|\partial X_i| = 0$ for all $i \in \mathcal{A}$. Similar results hold for X'_i.

Definition

We call $\cup_{j\in\mathcal{A}} X_j$ in $\mathcal{T}$ by *the center pieces* and if there exists a common translation $\pi(\mathbf{p}) \neq \mathbf{0}$ so that $X_j + \pi(\mathbf{p}) \in \mathcal{T}$ for every $j \in \mathcal{A}$, then we call $\cup_{j\in\mathcal{A}} X_j + \pi(\mathbf{p})$ by *satellite pieces*.

To make the argument simple, assume

Standing hypothesis

For $i \in \mathcal{A}$ and a finite word $W \in \mathcal{L}(u), W \neq i$, either $X_i + \pi(W) \in \mathcal{T}$ for some $n > 0$ or there is some satellite $X + \pi(\mathbf{p})$ so that $X_i + \pi(\mathbf{p}) + \pi(W) \in \mathcal{T}$ for some $n > 0$.

Definition

X_i and X_j *have contact face at* x if $|X_i \cap X_j| = 0$ and $x \in X_i \cap X_j$.
More generally, $E_0(\sigma)^n X_i + \pi(\mathbf{x})$ and $E_0(\sigma)^n X_j + \pi(\mathbf{y})$ *have contact face at* x if $|E_0(\sigma)^n X_i + \pi(\mathbf{x}) \cap E_0(\sigma)^n X_j + \pi(\mathbf{y})| = 0$ and $x \in E_0(\sigma)^n X_i + \pi(\mathbf{x}) \cap E_0(\sigma)^n X_j + \pi(\mathbf{y})$.

Suppose that X_i and X_j have contact face at x. Then there exist a sequence of pairs

$$X_i \supset E_0(\sigma) X_{i_1} + \pi(f(P^{(i_1)}_{I_1,1})) \supset \cdots \supset E_0(\sigma)^n X_{i_n} + \pi(f(P^{(i_n)}_{I_n,n})) \ni x,$$

$$X_j \supset E_0(\sigma) X_{j_1} + \pi(f(P^{(j_1)}_{J_1,1})) \supset \cdots \supset E_0(\sigma)^n X_{j_n} + \pi(f(P^{(j_n)}_{J_n,n})) \ni x,$$

which have contact face at x at each level. Notice that $\sigma^n(i_n)$ is a factor of

A direct consequence of the hypothesis:

Proposition

$T(X'_i \cap X'_j) \subset \partial X$ *and* $T^{-1}(X_i \cap X_j) \subset \partial X$ *for* $i \neq j$,

Boundary hopping

Definition

We say that $x \in X$ is *a point of finite type* if x has the form $\pi(f(u_0u_1 \cdots u_k))$ for some k.

The hypothesis guarantees that *points of finite type are preserved by any transformation generated by* $\pi(W), W \in \mathcal{L}(u)$, whereas there are many possibilities of way of tilings.

Boundary hopping

Proposition

Let $a \in X_i \cap X_j$. *Suppose that there is an orbit on the boundary*

$$\xi_k \in T^{-k} a \cap \partial X \quad (1 \leq k \leq n-1)$$

which starts with $\xi_1 = a + \pi(\mathbf{e}_j)$. *Let* $E_0(\sigma)^m X_{j_m} + \pi(f(P^{(j_m)}_{J_m,m})) \subset X_j$ *and* $E_0(\sigma)^m X_{i_m} + \pi(f(P^{(i_m)}_{I_m,m})) \subset X_i$ *have contact face at* a *for sufficiently large* m. *Suppose also that* j *of* $\sigma^m(j_m)$ *is realised at* u_k *and* i *of* $\sigma^m(i_m)$ *at* u_l.

$$u = u_0 u_1 \cdots \underbrace{\cdots \overset{k\text{th}}{j} \cdots}_{\sigma^m(j_m)} \cdots \underbrace{\cdots \overset{l\text{th}}{i} \cdots}_{\sigma^m(i_m)} \cdots$$

Then $T^{-1}\xi_{n-1} \in \mathrm{int}X$ *or* $\in \partial X \cap X_h \cap X_{h'}$ *for* $h \neq h'$ *if and only if* $|u_{[k,k+n-1]}|_i = |u_{[l,l+n-1]}|_i$ *for all* $i \in \mathcal{A}$.

Boundary hopping

We state the easiest case: suppose

$$\sigma^m(j_m) = \cdots jA_1A_2\cdots A_{n-1}\cdots, \sigma^m(i_m) = \cdots iB_1B_2\cdots B_{n-1}\cdots .$$

Then

$$\{j, A_1, \cdots, A_{n-1}\} = \{i, B_1, \cdots, B_{n-1}\}$$

if and only if

$$T^{-1}\xi_{n-1} \in \mathrm{int}X \text{ or } \in \partial X \cap X_h \cap X_{h'} .$$

Boundary identification

Proposition

$\exists n_0 > 0$ s.t. $\forall \xi \in \partial X$ jumps into the interior $\mathrm{int}X$ by up to n_0 times iterations of T. Similar results hold for T^{-1}.

Now it is clear that we can identify the boundary ∂X through interior points. Let us introduce a finite partition $\mathfrak{P}$ of ∂X. Set

$$P^{(1)} = \{\xi \in \partial X : \exists i, j(i \neq j) \text{ such that } T\xi \in \mathrm{int}X \cap X_i \cap X_j\}$$

and inductively define

$$P^{(k)} = \{\xi \in \partial X \backslash P^{(1)} \cup \cdots \cup P^{(k-1)} : T\xi \in P^{(k-1)}\}.$$

Prososition above implies that $\mathfrak{P} = \{P^{(k)}\}_{k=1}^{n_0}$ forms a disjoint covering of ∂X.

Definition

We say that $x, y \in \partial X$ can be 'dynamically' identified if we can find such a pair of subiles X_i', X_j' with contact face at a that

$$x, y \in T^n a \quad \text{for some } n > 0.$$

We can introduce a finite partition of ∂X through a local tiling. If $x, y \in \partial X$ can be 'dynamically' identified, then

$$x - y \in \Gamma_0.$$

Fundamental domain?

Lemma (Ito-Arnoux)

There exists an integer $l \geq 1$ such that the number of preimages $p^{-1}(\cdot)$ is l a.e.

We see X as "l-tiles of a fundamental domain" .
NOTICE THAT PISOT CONJECTURE IS
EQUIVALENT TO $l = 1$.
Might introduce a pesudo-distance just as in differential geometry. but WE DO NOT KNOW that $\tilde{X}$ is even arcwise-connected.

How to connect x and y in X

Let

$$[x, \gamma(y)]_t := x + t(\gamma(y) - x) \quad (0 \le t \le 1).$$

Let $C(t)$ be a piecewise smooth curve in $\mathcal{P}$ parametrized by $t \in [0, 1]$. The lift $p^{-1}p(C(t))$ consists of $\gamma(C(t))$ for $\gamma \in \Gamma_0$. Every pair, $\gamma_1(C(t))$ and $\gamma_2(C(t))$, translates to each other by Γ_0.

Proposition

After a finite number of special translations of Γ_0(boundary identification), any point ξ of $[x, \gamma(y)]_t$ enters into X. In particular, $\gamma(y)$ is translated into $\gamma' \circ \gamma(y)$ for some $\gamma' \in \Gamma_0$(folding process).

Definition

We say that $x, y \in X$ are joined in X if we can connect x and y through the boundary identification.

Definition

For $x, y \in X$, define

$$d_*(x, y) = \inf\{|x - \gamma(y)|\}$$

where infimum is taken over any $\gamma \in \Gamma_0$ by which x and y are joined in X.

Remark

Definitely x and y are joined by $[x, y]_t$ with $\gamma = id$. So we can take infimum.

Lemma

d_* is a pseudo-distance on $\tilde{X}$.

Remark

If $\#p^{-1}(\cdot) = 1$ a.e., which we are aiming at, then the definition above coincides with the one for the standard pseudo-distance d^*.

Our pseudo-distance d_* is *locally Euclidean*.

Proposition

Suppose $x \in \mathrm{int}\tilde{X}$. Then there exists $\epsilon_0 > 0$ so that if $d_*(x, y) \leq \epsilon$ for any $0 \leq \epsilon < \epsilon_0$, then $d_*(x, y) = |x - y|$. In particular, $d_*(x, y) > 0$ if $x \neq y$.

In summary, we obtain (P2).

Proposition

$d_*(x,y) = 0$ if and only if $x = y$ or $x, y \in \partial X$ can be dynamically identified.

We finally verify (P3). We shall show that the domain exchange flow $(\tilde{X}, \tilde{T})$ inherits minimality from the substitutive dynamical system $(\overline{O(u)}, S)$.

Lemma

$\tilde{X}$ is compact.

[Proof.] Since $(\tilde{X}, d)$ is a metric space, we may verify its sequential compactness.

Proposition

The domain exchange flow $(\tilde{X}, \tilde{T})$ is minimal.

Proof. By [Ito-Arnoux] it is proved that (X, T, ν) is measure-theoretically isomorphic to $(\overline{O(u)}, S, \mu)$ via ϕ. So there exist $\Omega_1 \subset \Omega, \nu(\Omega_1) = 1$ and $X_1 \subset X, \nu(X_1) = 1$ so that $S\Omega_1 \subset \Omega_1, TX_1 \subset X_1$ and $\phi : X_1 \to \Omega_1$ is bijective with $S \circ \phi = \phi \circ T$. Since $(\overline{O(u)}, S, \mu)$ is minimal, it follows that the orbit of almost every point of X is dense. Denote such points of X by Y. Thus $(\tilde{X}, \tilde{T})$ is topologically transitive. Since the domain exchange flow is an isometry, this concludes the proof.