Pisot Conjecture and Rauzy fractals

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Substitution

We may set $\mathcal{A} = \{1, 2, \cdots, d\}$. Denote by $\mathcal{A}^*$ the set of all finite words over $\mathcal{A}$.

A substitution (over $\mathcal{A}$) is a mapping $\sigma : \mathcal{A} \to \mathcal{A}^*$.

- $\sigma$ extends to $\mathcal{A}^*$ by requiring $\sigma(w_1w_2) = \sigma(w_1)\sigma(w_2)$.
- A substitution $\sigma$ is primitive if $\exists n > 0$ so that for $\forall i, j \in \mathcal{A}$ the sequence $\sigma^n(i)$ contains $j$.
- A substitution $\sigma$ naturally induces a mapping on $\mathcal{A}_{\mathbb{N}}$:

$$\sigma(u) := \sigma(u_0)\sigma(u_1)\cdots \text{ if } u = (u_k)_{k \in \mathbb{N}} \in \mathcal{A}_{\mathbb{N}}.$$  

By abuse of language, we use the same symbol $\sigma$ even when we let $\sigma$ act on $\mathcal{A}_{\mathbb{N}}$. 
Substitution dynamical system

Suppose that a primitive substitution \( \sigma \) has a fixed point \( u: \sigma(u) = u \).

Example (Morse-Tue substitution)
For \( \sigma(1) = 12, \sigma(2) = 21 \),

\[
    u = \lim_{n \to \infty} \sigma^n(1) = 1221211221121221221 \cdots
\]

is a fixed point.

WE WANT TO STUDY RECURRENCE PROPERTIES OF \( u \).

\[
    \overline{O(u)} = \{ S^n u : \forall n \geq 0 \} \subset A^\mathbb{N}.
\]

The subshift \((\overline{O(u)}, S)\) is called the substitution dynamical system for \( \sigma \).

- \((\overline{O(u)}, S)\) is minimal.
- \( \exists \nu \) a unique \( S \)-invariant ergodic measure on \( \overline{O(u)} \).
Spectrum

For a measure-preserving system \((O(u), S, \nu)\), define the unitary operator
\[ U_S : L^2(\nu) \to L^2(\nu) \]
by
\[ (U_S f)(x) := f(Sx). \]

- A complex number \(\lambda\) is an eigenvalue of \(S\) if \(\exists f \in L^2(\nu)\) non-zero such that \(U_S f = \lambda f\). This \(f\) is an eigenfunction of \(S\) corresponding to \(\lambda\).
- We say that the substitution dynamical system \((O(u), S, \nu)\) has discrete spectrum if there exist eigenfunctions of \(S\) which forms an orthonormal basis for \(L^2(\nu)\).

**Theorem (Halmos)**

*If an ergodic measure-preserving system has discrete spectrum, then it is isomorphic to some ergodic rotation on a compact abelian group.*
Unlike hyperbolic dynamical systems,

**Theorem (1978 Dekking-Keane)**

*The substitution dynamical system for a primitive substitution is never strongly mixing.*

The spectral type varies from the weakly mixing one to rotation, depending on the substitution.
Let $E_0(\sigma) : \mathbb{R}^d \to \mathbb{R}^d$ be a linear mapping, sometimes called *abelianization*, defined by

$$E_0(\sigma) = [f(\sigma(1)), f(\sigma(2)), \cdots, f(\sigma(d))]$$

where $f : \mathcal{A}^* \to \mathbb{Z}^d$ is a homomorphism defined by

$$f(w_1 w_2 \cdots w_k) = e_{w_1} + e_{w_2} + \cdots + e_{w_k}$$

with $f \circ \sigma = E_0(\sigma) \circ f$.

Primitivity is equivalent to the fact that $E_0(\sigma)^n$ is positive.
Pisot substitution

An algebraic integer $\lambda > 1$ is a Pisot(-Vijayaraghavan) number if the other roots are less than one in modulus.

Example

$\lambda^2 - \lambda - 1 = 0$ (golden mean), $\lambda^3 - \lambda^2 - \lambda - 1 = 0$ (tribonacci), etc.

A primitive substitution $\sigma$ is an irreducible Pisot substitution if the characteristic polynomial of $E_0(\sigma)$ is irreducible and its maximal eigenvalue is a Pisot number.
To study the spectrum, we may assume that any irreducible Pisot substitution has a fixed point.

**Pisot Conjecture**

If $\sigma$ is a (unimodlar), irreducible Pisot substitution, then the substitution dynamical system for $\sigma$ has discrete spectrum.

— Classification problem of recurrent sequences in ergodic theory
— Solved for $\#A = d = 2$ (2003 Hollander-Solomyak+Barge-Diamond)
— Solved for special cases of $d \geq 3$
Geometric representation of the substitution dynamical system

- $(d = 2)$ "cut & projection method" (folklore).
  $\exists$ 2-interval exchange transformation isomorphic to the substitution dynamical system for an irreducible Pisot substitution $\sigma$.

- $(d = 3)$ 1982 Rauzy’s breakthrough
  $\sigma_R(1) = 12, \sigma_R(2) = 13, \sigma_R(3) = 1$ (Rauzy substitution)

- $(d \geq 3)$ 2001 Ito-Arnoux generalized Rauzy’s framework into unimodular, irreducible Pisot substitutions with a general combinatorial condition.
Rauzy-Ito-Arnoux framework

Let $\mathcal{P}$ be the orthogonal subspace to a positive eigenvector of $tE_0(\sigma)$:

$E_0(\sigma)\mathcal{P} = \mathcal{P}$.

Let $\pi: \mathcal{P} \rightarrow \mathbb{R}^d$ be the projection along the maximal eigenvector for $E_0(\sigma)$. Denote the fixed point of $\sigma$ by $u = u_0u_1 \cdots u_k \cdots$.

Define Rauzy fractal $X$ (for $\sigma$) by

$$X = \{ \pi f(u_0u_1 \cdots u_k) | k = 0, 1, 2, \cdots \} = \{ \pi \sum_{j=0}^{k} e_{u_j} | k = 0, 1, 2, \cdots \}$$

Set also

$$X'_i = \{ \pi f(u_0u_1 \cdots u_k) | u_k = i \text{ for some } k \}$$

and

$$X_i = \{ \pi f(u_0u_1 \cdots u_{k-1}) | u_k = i \text{ for some } k \}$$

for $1 \leq i \leq d$. 
Then

\[ X = \bigcup_{i \in \mathcal{A}} X'_i = \bigcup_{i \in \mathcal{A}} X_i. \]

These two partitions are related to each other by a translation

\[ X'_i - \pi(e_i) = X_i \quad \text{for } 1 \leq i \leq d. \]

Define the **domain exchange transformation** \( T : X \rightarrow X \) by

\[ T : X'_i \rightarrow X_i, \quad T(x) = x - \pi(e_i) \]

and its inverse transformation by

\[ T^{-1} : X_i \rightarrow X'_i, \quad T^{-1}(x) = x + \pi(e_i). \]
Let $\Gamma_0$ be the **discrete subgroup** generated by translations

$$\pi(e_i - e_j) \quad (1 \leq i < j \leq d)$$

acting on $P$.

Every discrete group of isometries of a metric space is properly discontinuous. ($\Rightarrow$ fundamental domain)

Denote the canonical projection by $p : P \rightarrow P/\Gamma_0$ and the quotient space $P/\Gamma_0$ by $\mathbb{T}^{d-1}$.
Rauzy’s case

Lebesgue measure on $X$ is $T$-invariant.

$(O(u), S, \nu)$ is isomorphic to $(X, T, m)$ where $m$ is the normalised Lebesgue measure on $X$.

$(T, X)$ singly covers a minimal translation on 2-torus (Rauzy fractal $X$ is the closure of a ”fractal” fundamental domain).

Any ergodic rotation w.r.t Haar measure has discrete spectrum. $\Rightarrow$ Pisot conjecture holds for this case.
When we pay attention to a particular letter $w_k^{(j)}$ in $\sigma^n(j)$, we write
$$\sigma^n(j) = P_{k,n}^{(j)} w_k^{(j)} S_{k,n}^{(j)},$$
where $P_{k,n}^{(j)}$ and $S_{k,n}^{(j)}$ represent its prefix and suffix respectively.

**Definition (Strong coincidence condition)**

A substitution $\sigma$ has a (positive) strong coincidence of order 1 for $i, j \in A$ if there exists $k > 0$ such that $w_k^{(i)} = w_k^{(j)}$ and $f(P_{k,1}^{(i)}) = f(P_{k,1}^{(j)})$. It has a (positive) strong coincidence for $i, j \in A$ if, for some $n > 0$, $\sigma^n$ has a strong coincidence of order 1 for $i, j \in A$. A substitution $\sigma$ has a (positive) strong coincidence for all letters if it has a strong coincidence for any pair $i, j \in A$. 

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Theorem (2001 Arnoux-Ito)

Let $\sigma$ be an irreducible unimodular Pisot substitution over $\mathbb{A}$. If $\sigma$ has a strong coincidence, then there exists a dynamical system $(X, T, \nu)$, the domain exchange system, so that it is measure-theoretically isomorphic to $(\overline{O(u)}, S, \nu)$ and is semi-conjugate to a minimal translation on the torus $(\mathbb{T}^{d-1}, \tau)$. Moreover $(X, T)$ is a finite extension of $(\mathbb{T}^{d-1}, \tau)$.
Theorem (N)

*Pisot conjecture is true for any irreducible, unimodular Pisot substitution with strong coincidence condition.*
The domain exchange transformations generate the discrete group. But the converse is NOT true!

\[ T(x) = x - \pi(e_i), \quad \Gamma_0 = \left\{ x + \sum c_{ij} \pi(e_i - e_j) : c_{ij} \in \mathbb{Z} \right\}. \]

It may not be enough to take the quotient by \( \Gamma_0 \) to see the structure of the substitution dynamical system in Rauzy fractals.

Thus, we construct a discrete flow isomorphic to the domain exchange system instead of the toral translation.
Pseudo-distance on Rauzy fractals

Remember that a function $d_* : X \times X \to \mathbb{R}$ is called a pseudo-distance if it satisfies that for $\forall x, y, z \in X$

1. $d_*(x, y) \geq 0$, 2. $x = y \Rightarrow d_*(x, y) = 0$,

3. $d_*(x, y) = d_*(y, x)$, 4. $d_*(x, y) \leq d_*(x, z) + d_*(z, y)$.

Notice that if you replace "$\Rightarrow$" by "$\Leftrightarrow$" in (2), then $d_*$ is called a distance on $X$.

If you have a pseudo-distance $d_*$ on $X$, you can make a metric space from $X$(General topology):
Define an equivalent relation on $X$ by

$$x \sim y \iff d_*(x, y) = 0.$$ 

Denote the quotient space by $\tilde{X} = X \setminus \sim$. Then you can define the distance $d$ on $\tilde{X}$ by

$$d([x], [y]) := d_*(x, y) \quad \forall [x], [y] \in \tilde{X}.$$
Example

\[ \mathbb{T}^2 = [0, 1]^2 / \sim \]

How to introduce the distance on 2-torus from a pseudo-distance:

\[ d^*(x, y) = \min \{|x - \gamma(y)| : \gamma \in \Gamma_0\} \quad (x, y \in [0, 1]^2). \]

where \( \Gamma_0 = \langle \gamma_1, \gamma_2 \rangle \) and

\[ \gamma_1(a, b) = (a + 1, b), \quad \gamma_2(a, b) = (a, b + 1). \]

\[ \Rightarrow \quad d^*(x, y) = 0 \iff x \sim y \text{ (identification on the boundary)} \]
FOR THE MOMENT, SUPPOSE THAT WE HAVE SUCH A PSEUDO-DISTANCE $d_*$ ON RAUZY FRACTAL $X$ THAT

(P1) (isometry) $d_*(Tx, Ty) = d_*(x, y)$,

(P2) $d_*(x, y) = 0 \iff x = y$ or $x, y \in \partial X$ with $x - y \in \Gamma_0$.

(P3) The quotient $\tilde{X}$ is compact and $\tilde{T} : \tilde{X} \rightarrow \tilde{X}$ by $\tilde{T}[x] := [Tx]$ is minimal.
(\tilde{X}, d) \text{ is a compact metric space.}

Define \tilde{U} : \tilde{X} \to \tilde{X} by \tilde{U}[x] := [T^{-1}x]. It is easy to see that \tilde{T}\tilde{U} = \tilde{U}\tilde{T} = \text{id}_{\tilde{X}}. Thus \tilde{T} is invertible.

(P1) implies that \tilde{T} is an isometry: d(\tilde{T}[x], \tilde{T}[y]) = d([x], [y]). \Rightarrow (\tilde{X}, \tilde{T}) \text{ is a homeomorphism on a compact metric space.}

We call (\tilde{X}, \tilde{T}) \text{ the domain exchange flow} to emphasize its invertibility.
Theorem (Halmos-von Neumann)

Let $\tilde{T} : \tilde{X} \to \tilde{X}$ be a homeomorphism on a compact metric space. The followings are equivalent.

(i) $\tilde{T}$ is topologically transitive ($\exists [x_0] \in \tilde{X} \mathcal{O}([x_0]) = \{\tilde{T}^n[x_0]\}_{n \in \mathbb{Z}}$ is dense in $\tilde{X}$) and is an isometry for some metric $d$ on $\tilde{X}$

(ii) $\tilde{T}$ is topologically conjugate to a minimal rotation on a compact abelian metric group.

A group structure on $\tilde{X}$ can be introduced through $\mathcal{O}([x_0])$: a multiplication $\ast$ is defined by $\tilde{T}^n[x_0] \ast \tilde{T}^m[x_0] := \tilde{T}^{n+m}[x_0]$. The isometry $d(\tilde{T}[x], \tilde{T}[y]) = d([x], [y])$ allows group operations to extend on $\tilde{X}$. For any $[x] \in \tilde{X}$, $\tilde{T}[x] = \lim_{n \to \infty} \tilde{T}\tilde{T}^n[x_0] = \lim_{n \to \infty} \tilde{T}[x_0] \ast \tilde{T}^n[x_0] = (\tilde{T}[x_0]) \ast [x]$, thus a rotation.
Idea to prove Pisot conjecture

(P1), (P2) and (P3) guarantees that $\tilde{T}$ satisfies (i) of Halmos-von Neumann theorem. So $\tilde{T}$ is conjugate to a minimal rotation on a compact abelian group.

Let $G$ be a compact topological group. Then there exists a unique probability measure $m$ on $G$ so that

$$m(xE) = m(E) \quad \forall x \in G \quad \forall E \in \mathcal{B}(G).$$

This measure is called the normalised Haar measure. The $T$-invariant ergodic measure $\nu$ on $X$ induces a $\tilde{T}$-invariant measure $\mu$ on $\tilde{X}$: $\mu = \nu \circ q^{-1}$ where $q : X \to \tilde{X}$, $q(x) = [x]$.

We can prove

**Lemma**

The induced invariant measure $\mu = \nu \circ q^{-1}$ is the normalised Haar measure on $\tilde{X}$. 
(P2) implies that the difference between $\tilde{X}$ and $X$ occurs only on the boundary $\partial X$ and the interior $\text{int}\ X$ is preserved by $q$. Since $\nu(\partial X) = 0$, we conclude

**Lemma**

$(\tilde{X}, \tilde{T}, \mu)$ is measure-theoretically isomorphic to $(X, T, \nu)$.

\[
\begin{array}{ccc}
(X, \nu) & \xrightarrow{T} & (X, \nu) \\
\downarrow q & & \downarrow q \\
(\tilde{X}, \mu) & \xrightarrow{\tilde{T}} & (\tilde{X}, \mu)
\end{array}
\]
We refer to the following two standard theorems in Ergodic theory.

**Theorem**

Let $G$ be a compact group and $\tilde{T}(g) = ag$ be a rotation of $G$. Then $\tilde{T}$ is ergodic w.r.t the normalised Haar measure iff $\{a^n\}_{n \in \mathbb{Z}}$ is dense in $G$.

**Theorem**

Let $\tilde{T}(g) = ag$ be an ergodic rotation of a compact abelian group $G$. Then $\tilde{T}$ has discrete spectrum.
Consequently, $(\tilde{X}, \tilde{T}, \mu)$ has discrete spectrum and so do $(X, T, \nu)$ and $(O(u), S, \nu)$.
Which proves Pisot conjecture for irreducible, unimodular Pisot substitutions with strong coincidence.
How to construct such a nice pseudo-distance

We hope to construct a similar pseudo-distance on Rauzy fractals that identifies the boundary just as we see in the case of 2-torus.

Difficulties:

- How to identify the fractal boundary?
- No guarantee that $X$ is the closure of a fundamental domain.
Idea to prove Pisot conjecture

Theorem (Set equation)

Given $i \in \mathcal{A}$ and $n \geq 1$, the following set equation holds:

$$X'_i = \sum_{j \in \mathcal{A}} \sum_{\sigma^n(j) = P_{k,n}^{(j)} i S_{k,n}^{(j)}} E_0(\sigma)^n X'_j - \pi(f(S_{k,n}^{(j)})).$$

Similarly

$$X_i = \sum_{j \in \mathcal{A}} \sum_{\sigma^n(j) = P_{k,n}^{(j)} i S_{k,n}^{(j)}} E_0(\sigma)^n X_j + \pi(f(P_{k,n}^{(j)})).$$
Idea to prove Pisot conjecture

From the set equation, $E_0(\sigma)^{-n}X$ makes a local multi-tiling $\mathcal{T}$ on $\mathcal{P}$ for each $n \geq 0$:

$$\mathcal{T} = \{ X_j + \pi(f(P^{(i)}_{k,n})) : \forall j \in \mathcal{A}, \forall i \in \mathcal{A}, \forall k \geq 1, \sigma^n(i) = P^{(i)}_{k,n}jS^{(i)}_{k,n} \}.$$ 

Theorem (Arnoux-Ito,Feng-Furukado-Ito-Wu)

Under the same assumption of [Arnoux-Ito], $\mathcal{T}$ is a local tiling. In other words, $|X_i + \pi(p) \cap X_j + \pi(q)| = 0$ for any distinct pair $X_i + \pi(p), X_j + \pi(q) \in \mathcal{T}$. Furthermore, $|\partial X_i| = 0$ for all $i \in \mathcal{A}$. Similar results hold for $X'_i$.

Definition

We call $\bigcup_{j \in \mathcal{A}} X_j$ in $\mathcal{T}$ by the center pieces and if there exists a common translation $\pi(p) \neq 0$ so that $X_j + \pi(p) \in \mathcal{T}$ for every $j \in \mathcal{A}$, then we call $\bigcup_{j \in \mathcal{A}} X_j + \pi(p)$ by satellite pieces.
To make the argument simple, assume

**Standing hypothesis**

For $i \in A$ and a finite word $W \in \mathcal{L}(u)$, $W \neq i$, either $X_i + \pi(W) \in \mathcal{T}$ for some $n > 0$ or there is some satellite $X + \pi(p)$ so that $X_i + \pi(p) + \pi(W) \in \mathcal{T}$ for some $n > 0$.

**Definition**

$X_i$ and $X_j$ have contact face at $x$ if $|X_i \cap X_j| = 0$ and $x \in X_i \cap X_j$.

More generally, $E_0(\sigma)^nX_i + \pi(x)$ and $E_0(\sigma)^nX_j + \pi(y)$ have contact face at $x$ if $|E_0(\sigma)^nX_i + \pi(x) \cap E_0(\sigma)^nX_j + \pi(y)| = 0$ and $x \in E_0(\sigma)^nX_i + \pi(x) \cap E_0(\sigma)^nX_j + \pi(y)$.

Suppose that $X_i$ and $X_j$ have contact face at $x$. Then there exist a sequence of pairs

$$X_i \supset E_0(\sigma)X_{i_1} + \pi(f(P_{I_1,1}^{(i_1)})) \supset \cdots \supset E_0(\sigma)^nX_{i_n} + \pi(f(P_{I_n,1}^{(i_n)})) \ni x,$$

$$X_j \supset E_0(\sigma)X_{j_1} + \pi(f(P_{J_1,1}^{(j_1)})) \supset \cdots \supset E_0(\sigma)^nX_{j_n} + \pi(f(P_{J_n,1}^{(j_n)})) \ni x,$$

which have contact face at $x$ at each level. Notice that $\sigma^n(i_n)$ is a factor of
A direct consequence of the hypothesis:

**Proposition**

\[ T(X'_i \cap X'_j) \subset \partial X \text{ and } T^{-1}(X_i \cap X_j) \subset \partial X \text{ for } i \neq j, \]
Definition

We say that $x \in X$ is a point of finite type if $x$ has the form $\pi(f(u_0 u_1 \cdots u_k))$ for some $k$.

The hypothesis guarantees that points of finite type are preserved by any transformation generated by $\pi(W)$, $W \in \mathcal{L}(u)$, whereas there are many possibilities of way of tilings.
Idea to prove Pisot conjecture

Boundary hopping

Proposition

Let \( a \in X_i \cap X_j \). Suppose that there is an orbit on the boundary

\[ \xi_k \in T^{-k}a \cap \partial X \quad (1 \leq k \leq n - 1) \]

which starts with \( \xi_1 = a + \pi(e_j) \). Let \( E_0(\sigma)^mX_{jm} + \pi(f(P_{jm,m}^{(j)}) \subset X_j \) and \( E_0(\sigma)^mX_{im} + \pi(f(P_{im,m}^{(i)}) \subset X_i \) have contact face at \( a \) for sufficiently large \( m \). Suppose also that \( j \) of \( \sigma^m(j_m) \) is realised at \( u_k \) and \( i \) of \( \sigma^m(i_m) \) at \( u_l \).

\[
u = \nu_0 \nu_1 \cdots \underbrace{\nu_j \cdots \nu_i}_{\sigma^m(j_m)} \cdots \underbrace{\nu_l \cdots \nu_i}_{\sigma^m(i_m)}
\]

Then \( T^{-1}\xi_{n-1} \in \text{int}X \) or \( \in \partial X \cap X_h \cap X_{h'} \) for \( h \neq h' \) if and only if

\[ |u_{[k,k+n-1]}|_i = |u_{[l,l+n-1]}|_i \] for all \( i \in \mathcal{A} \).
Boundary hopping

We state the easiest case: suppose

$$\sigma^m(j_m) = \cdots jA_1A_2 \cdots A_{n-1} \cdots, \sigma^m(i_m) = \cdots iB_1B_2 \cdots B_{n-1} \cdots.$$ 

Then

$$\{j, A_1, \cdots, A_{n-1}\} = \{i, B_1, \cdots, B_{n-1}\}$$

if and only if

$$T^{-1} \xi_{n-1} \in \text{int}X \text{ or } \in \partial X \cap X_h \cap X_{h'}.$$
Idea to prove Pisot conjecture

Boundary identification

Proposition

\[ \exists n_0 > 0 \text{ s.t. } \forall \xi \in \partial X \text{ jumps into the interior } \text{int} X \text{ by up to } n_0 \text{ times iterations of } T. \text{ Similar results hold for } T^{-1}. \]

Now it is clear that we can identify the boundary \( \partial X \) through interior points. Let us introduce a finite partition \( \mathcal{P} \) of \( \partial X \). Set

\[ P^{(1)} = \{ \xi \in \partial X : \exists i, j (i \neq j) \text{ such that } T\xi \in \text{int} X \cap X_i \cap X_j \} \]

and inductively define

\[ P^{(k)} = \{ \xi \in \partial X \setminus P^{(1)} \cup \cdots \cup P^{(k-1)} : T\xi \in P^{(k-1)} \}. \]

Proposition above implies that \( \mathcal{P} = \{ P^{(k)} \}_{k=1}^{n_0} \) forms a disjoint covering of \( \partial X \).
**Definition**

We say that $x, y \in \partial X$ can be ’dynamically’ identified if we can find such a pair of subiles $X_i', X_j'$ with contact face at $a$ that

$$x, y \in T^n a \quad \text{for some } n > 0.$$ 

We can introduce a finite partition of $\partial X$ through a local tiling. If $x, y \in \partial X$ can be ’dynamically’ identified, then

$$x - y \in \Gamma_0.$$
Fundamental domain?

Lemma (Ito-Arnoux)

There exists an integer $l \geq 1$ such that the number of preimages $p^{-1}(\cdot)$ is $l$ a.e.

We see $X$ as ”$l$-tiles of a fundamental domain”. NOTICE THAT PISOT CONJECTURE IS EQUIVALENT TO $l = 1$.

Might introduce a pseudo-distance just as in differential geometry. but WE DO NOT KNOW that $\tilde{X}$ is even arcwise-connected.
How to connect $x$ and $y$ in $X$

Let

$$[x, \gamma(y)]_t := x + t(\gamma(y) - x) \quad (0 \leq t \leq 1).$$

Let $C(t)$ be a piecewise smooth curve in $\mathcal{P}$ parametrized by $t \in [0, 1]$. The lift $p^{-1}p(C(t))$ consists of $\gamma(C(t))$ for $\gamma \in \Gamma_0$. Every pair, $\gamma_1(C(t))$ and $\gamma_2(C(t))$, translates to each other by $\Gamma_0$.

**Proposition**

*After a finite number of special translations of $\Gamma_0$ (boundary identification), any point $\xi$ of $[x, \gamma(y)]_t$ enters into $X$. In particular, $\gamma(y)$ is translated into $\gamma' \circ \gamma(y)$ for some $\gamma' \in \Gamma_0$ (folding process).*
Definition
We say that $x, y \in X$ are joined in $X$ if we can connect $x$ and $y$ through the boundary identification.

Definition
For $x, y \in X$, define

$$d_*(x, y) = \inf \{|x - \gamma(y)|\}$$

where infimum is taken over any $\gamma \in \Gamma_0$ by which $x$ and $y$ are joined in $X$.

Remark
Definitely $x$ and $y$ are joined by $[x, y]_t$ with $\gamma = id$. So we can take infimum.
Lemma

$d_\ast$ is a pseudo-distance on $\tilde{X}$.

Remark

If $\#p^{-1}(\cdot) = 1$ a.e., which we are aiming at, then the definition above coincides with the one for the standard pseudo-distance $d^\ast$.

Our pseudo-distance $d_\ast$ is locally Euclidean.

Proposition

Suppose $x \in \text{int}\tilde{X}$. Then there exists $\epsilon_0 > 0$ so that if $d_\ast(x, y) \leq \epsilon$ for any $0 \leq \epsilon < \epsilon_0$, then $d_\ast(x, y) = |x - y|$. In particular, $d_\ast(x, y) > 0$ if $x \neq y$. 
In summary, we obtain (P2).

**Proposition**

\[ d_*(x, y) = 0 \text{ if and only if } x = y \text{ or } x, y \in \partial X \text{ can be dynamically identified.} \]
We finally verify (P3). We shall show that the domain exchange flow \((\tilde{X}, \tilde{T})\) inherits minimality from the substitutive dynamical system \((O(u), S)\).

**Lemma**

\(\tilde{X}\) is compact.

**Proof.** Since \((\tilde{X}, d)\) is a metric space, we may verify its sequential compactness.

**Proposition**

The domain exchange flow \((\tilde{X}, \tilde{T})\) is minimal.

Proof. By [Ito-Arnoux] it is proved that \((X, T, \nu)\) is measure-theoretically isomorphic to \((O(u), S, \mu)\) via \(\phi\). So there exist \(\Omega_1 \subset \Omega, \nu(\Omega_1) = 1\) and \(X_1 \subset X, \nu(X_1) = 1\) so that \(S\Omega_1 \subset \Omega_1, TX_1 \subset X_1\) and \(\phi : X_1 \to \Omega_1\) is bijective with \(S \circ \phi = \phi \circ T\). Since \((O(u), S, \mu)\) is minimal, it follows that the orbit of almost every point of \(X\) is dense. Denote such points of \(X\) by \(Y\). Thus \((\tilde{X}, \tilde{T})\) is topologically transitive. Since the domain exchange flow is an isometry, this concludes the proof.