Mapping Class Group, Rigidity, and Abstract Tiling Spaces

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Standing Hypotheses

A tiling $\mathcal{T}$, its tiling space $\mathcal{X}$, translation $\mathbb{R}^d$-action $T^t : \mathcal{X} \to \mathcal{X}$

- $\mathcal{T}$ is of finite complexity ($\equiv X$ is compact),
- $\mathcal{T}$ is repetitive ($\equiv T^t$ is minimal),
- $\mathcal{T}$ is ϕ-self-affine ($\implies \Phi \circ T^t = T^{\phi t} \circ \Phi$).
- $\mathcal{T}$ is aperiodic* ($\equiv T^t$ has no periods),

$*$ I have included aperiodicity hypothesis so that $\Phi : \mathcal{X} \to \mathcal{X}$ is a homeo, which I like.

Recap

We studied expansion factors ϕ (responsible for self-affinity).

Theorem

ϕ is an expansion factor $\implies$ ϕ is Perron integral algebraic.

Hopefully a Theorem

ϕ is Perron integral algebraic $\implies$ ϕ (or ϕ^n for some n) is an expansion factor (of some $\mathcal{T}$).

Simple Observation: If S is a symmetry of a tiling then $\phi_{\text{new}} := S \circ \phi$ is another expansion factor.

Perspective: Expansion factors of $\mathcal{T}$ are the expanding elements of the "general linear group" of $\mathcal{T}$.

Outline

General Linear Group of Tiling

Let $x_0 := \mathcal{T}$, a point in the **tiling space** $\mathcal{X} := \mathrm{cl}\{x_0 - t : \ t \in \mathbb{R}^d\}$.

Linear automorphisms (or **linear symmetries**) of a tiling x_0 are the homeomorphisms $h : \mathcal{X} \to \mathcal{X}$ with

$$\exists_{H \text{ linear isomorphism}} \quad h(x_0 + t) = x_0 + Ht \quad (\forall t \in \mathbb{R}^d).$$

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Definition

The **general linear group** *of a tiling* x_0 *is*

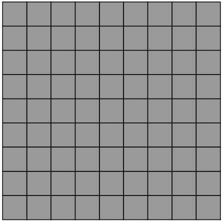
$$\mathbb{GL}(x_0) := \{H : \ H \text{ as above}\}.$$

Theorem (Easy*)

$\mathbb{GL}(x_0)$ *is a discrete subgroup of* $\mathbb{GL}(d, \mathbb{R})$.

* true for any minimal (LF) $\mathbb{R}^d$-action.

Example: Unit Square Tiling



Here $\mathcal{X} \simeq \mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$ and linear automorphisms are

$$h(x, y) = (ax + by, cx + dy)$$

where

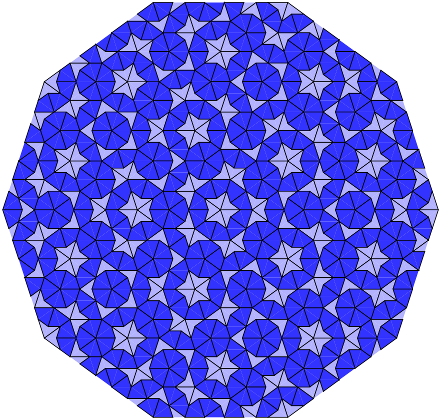
$$H := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{GL}(2, \mathbb{Z}).$$

So

$$\mathbb{GL}(\text{square tiling}) = \mathbb{GL}(2, \mathbb{Z}).$$

Linear Automorphisms of Penrose “Sun” Tiling

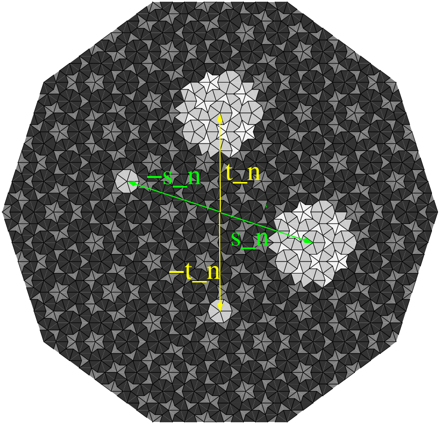
The Penrose Kite and Dart “Sun” Tiling



is visibly five fold symmetric and not centrally symmetric...

Linear Automorphisms of Penrose "Sun" Tiling

... yet it is topologically[1] centrally symmetric



$$i.e., \quad \mathrm{dist}(x_0 + t_n, x_0 + s_n) \to 0 \Leftrightarrow \mathrm{dist}(x_0 - t_n, x_0 - s_n) \to 0.$$

[1] Its statistical central symmetry was pointed out by Radin.

Linear Automorphisms Penrose "Sun" Tiling

Theorem (Symmetries of Penrose)

Let $\lambda = (\sqrt{5}+1)/2$.
The general linear group of the "sun" Penrose tiling is generated by

$$\Lambda = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \quad R_{2\pi/10} = \begin{bmatrix} \cos(2\pi/10) & \sin(2\pi/10) \\ -\sin(2\pi/10) & \cos(2\pi/10) \end{bmatrix}, \quad F = \begin{bmatrix} +1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Invitation

Systematically compute $\mathbb{GL}(x_0)$
(for tiling actions ... and other important minimal actions).

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NOTE: The def of $\mathbb{GL}(x_0)$ generalizes to any $\mathbb{R}^d$-action.

Vast Open Area: $\mathbb{GL}$(Kronecker) =???

Motivated by geometric realization $\mathcal{X} \to \mathbb{T}^d$:

Let $L \subset \mathbb{R}^n$ be a linear subspace. Describe the **group**

$$\mathcal{G}_L := \{A \in \mathbb{GL}_n(\mathbb{Z}) : A(L) \subset L\}$$

or at least the **algebra** of $n \times n$ matrices over $\mathbb{Q}$ given by

$$\mathcal{A}_L := \{A : \ A(L) \subset L\}.$$

NOTE: If L is algebraic (irred.) of $\dim(L) = 1$, then $\mathcal{A}_L$ is a finite extension field of $\mathbb{Q}$ and $\mathcal{G}_L$ is the group of its integral units.

Q: What if $d := \dim(L) > 1$?
A: (Sarnak) For $n > 3$ no systematic classification is known; look at examples and specific questions. *Need a rational algebraic geometer!*

Outline

Mapping Class Group of Tiling

Topological Automorphisms/Symmeteries of a tiling x_0 are the pointed homeomorphisms $h : \mathcal{X} \to \mathcal{X}$, $h(x_0) = x_0$. They form an infinite dimensional group $\mathcal{H}(\mathcal{X}, x_0)$ whose structure is governed by

Definition

The **Mapping Class Group** *of* x_0*:*

$$\mathcal{M}CG(\mathcal{X}, x_0) := \mathcal{H}(\mathcal{X}, x_0)/\mathcal{H}_0(\mathcal{X}, x_0) \qquad (1)$$

where $\mathcal{H}_0(\mathcal{X}, x_0)$ *are the homeomorphisms homotopic to the identity.*

Mapping Class Group Theorem

Extra Hypothesis*:
tiling x_0 is **self-similar**, i.e., ϕ-self-affine with conformal ϕ.

$\mathbb{GL}(x_0)$ captures all linear and non-linear maps in $\mathcal{H}(\mathcal{X}, x_0)$:

Theorem
Any topological symmetry is homotopic to a linear one so that $\mathcal{M}CG(\mathcal{X}, x_0)$ is (naturally) isomorphic to $\mathbb{GL}(x_0)$, which is a **discrete subgroup** *of $\mathbb{GL}_d(\mathbb{R})$.*

$*$ *Q: Is conformality really necessary?*

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Main Theorem (global linearization)

Loosely:

$$\text{(topology of } \mathcal{X}) \longrightarrow \text{(dynamics/analysis on } \mathcal{X})$$

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Precisely:

Theorem (Topological Rigidity Theorem)

If $\mathcal{X}$ and $\tilde{\mathcal{X}}$ are self-similar tiling spaces and there is a homeomorphism $h_0 : \mathcal{X} \to \tilde{\mathcal{X}}$, then there is a homeomorphism $h : X \to \tilde{\mathcal{X}}$ which is **linear (along the leaves)**, *i.e.,*

$$\exists_{H \text{ linear iso}} \quad h(x+t) = h(x) + Ht \quad (\forall t \in \mathbb{R}^d, x \in \mathcal{X}).$$

$$h \circ T^t = T^{Ht} \circ h.$$

Baby Example (of linearization)

Take $X = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, $x_0 = 0$, $T^t(x) := x + t$, then

$$\mathcal{M}CG(\mathbb{T}^2, 0) := \mathcal{H}(\mathbb{T}^2, 0)/\mathcal{H}_0(\mathbb{T}^2, 0) \simeq \mathbb{GL}(2, \mathbb{Z}).$$

For instance

$$h_0(x, y) = (x + y + \sin(2\pi x), x + \cos(2\pi y)) \ (mod 1)$$

"linearizes" to

$$h(x, y) = (x + y, x) \ (mod 1)$$

which corresponds to

$$H = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \in \mathbb{GL}(2, \mathbb{Z}).$$

An Analogy: Uniformization of Riemann Surfaces

For comparison, two familiar instances of "rigidity":

Theorem (Riemann Mapping Theorem)

If $\mathcal{X}$ and $\tilde{X}$ are two homeomorphic *simply connected proper domains in $\mathbb{C}$, then they are* biholomorphic*. (In fact, they are biholomorphic to the open unit disk.)*

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Theorem (Uniformization of Elliptic Surfaces)

If $\mathcal{X}$ and $\tilde{X}$ are two homeomorphic *Riemann surfaces of genus* 1 *then they are* biholomorphic *up to a linear deformation. (In fact, $\exists_{H \text{ linear iso}}$ they are biholomorphic to $\mathbb{R}^2/H\mathbb{Z}^2$.)*

Example Corollaries

Corollary (for classifiers)

Spectral invariants are topological invariants for self-similar tiling spaces.

Corollary (for physicists)

To determine the X-ray diffraction of x_0 (up to a linear transformation) it is enough to understand the topology of $\mathcal{X}$, not the geometry of x_0.

Outline

Bourbaki's Dream

Characterize the self-affine tiling spaces and their translation by concise axioms:

Issue: Somebody hands you an $\mathbb{R}^d$-action $T^t : X \to X$.

How can you tell that T "is" a self-affine tiling action?

That is,
When is there a self-affine, repetitive aperiodic tiling $\mathcal{T}$ of finite local complexity such that T is conjugate to the translation action on the tiling space $\mathcal{X}$ associated to $\mathcal{T}$?

NOTE: The less structure the axioms need, the better.

Topological Characterization Thm JK (2006 + K-Sobek 2013)

Let T be an $\mathbb{R}^d$-action on a compact metric space (X, d).
T is conjugate to a *self-affine tiling action* iff $\exists_{r_0, l_0, \Delta_0 > 0}$ so that

(EXP): $\exists \Phi : X \to X$ homeo and expanding $\Lambda : \mathbb{R}^d \to \mathbb{R}^d$ such that (writing $x + t$ for $T^t x$)

$$\Phi(x + t) = \Phi(x) + \Lambda t \qquad (\forall t \in \mathbb{R}^d,\ x \in X).$$

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(PS): $\forall_{0<\delta<\Delta_0}\ \exists_{\delta_1>0}\ \forall_{x,y \in X,\ U \subset \mathbb{R}^d \text{ connected}}$ if $\forall_{t \in U}\ \inf_{|s| \leq l_0} \mathrm{d}(x+t, y+s+t) < \delta_1$, then $\exists_{s \in B_{r_0}} \forall_{t \in U}\ \mathrm{d}(x+t, y+s+t) < \delta$.

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Actions as above are called **abstract self-affine tiling actions**.

Elements of Proof

1. **local product structure** from soft axioms; (rigidity?!)
2. existence of a **Markov partition** gives a tiling;
3. **zero dimensionality of the vertical X_0**. (new, see below)

Ad 3: Loosely:
" abstract self-affine tiling spaces have finite local complexity"

Theorem (Soft FLC K-Sobek, to be completed in 2013)
If T is an abstract self-affine tiling action, then X is locally a product of a Cantor and $\mathbb{R}^d$.

Idea behind Soft FLC Theorem

Think of the discrete action on the vertical X_0.

Step 1:

" expansivity implies expanding for $\mathbb{Z}^d$-actions"

Precisely:

Theorem (K-Sobek 2012)

For $d \in \mathbb{N}$ there are numbers $a, b > 0$ such that if T is an expansive $\mathbb{Z}^d$-action on a compact space X_0, then there exists a metric $\tilde{d}$ on X and numbers $\tilde{\beta} > 1$ and $\delta > 0$ so that, for all $x, y \in X_0$,

$$\max_{\mathbf{n} \in \mathbb{Z}^d,\, a < |\mathbf{n}| < b} \left\{ \tilde{d}\left(T^{\mathbf{n}} x, T^{\mathbf{n}} y\right) \right\} \geq \min \left\{ \tilde{\beta} \tilde{d}(x, y), \delta \right\}. \qquad (2)$$

Modeled on Fathi's result for $\mathbb{Z}$-actions.

Idea behind Soft FLC Theorem (cont.)

Step 2:

Theorem (K-Sobek 2012)

Let T be an expansive $\mathbb{Z}^d$-action on a compact metric space X_0. Then there b as in Theorem 2 we have

$$HD_{\tilde{d}}(X_0) \leq \frac{b}{\ln \tilde{\beta}} \cdot h_1(T) \tag{3}$$

where $h_1(T)$ is the **one-dimensional entropy** *of T, as given by*

$$h_1(T) := \lim_{\epsilon \to 0} \limsup_{R \to \infty} \frac{\ln \max \left\{ \#S : S \ ([-R, R]^d, \epsilon) - separated \right\}}{R^1}.$$

To finish: X_0 is Cantor since $h_1(T) = 0$ yields $HD_{\tilde{d}}(X_0) = 0$.