# Expansion factors for self-affine tilings
(Subtile 2013, Marseille)

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# Outline

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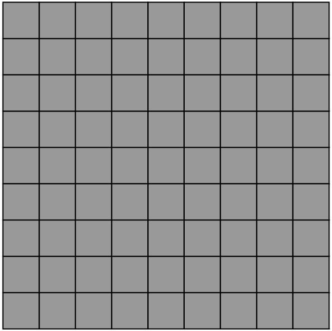
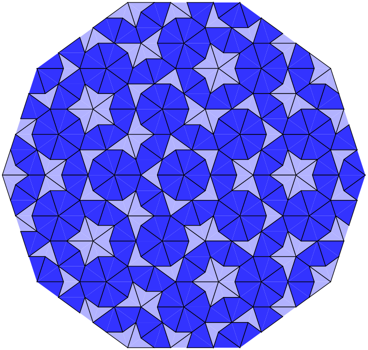
# Tilings of **Physical Space** $\mathbb{R}^d$

$\mathcal{T}$ **repetitive** tiling with **finite local complexity**





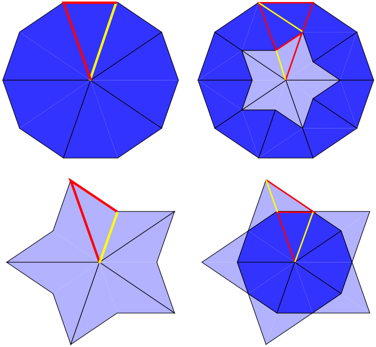
# **Expansion Factors** of Tilings

A linear expanding $\phi : \mathbb{R}^d \to \mathbb{R}^d$ is an **expansion factor** iff there is a $\phi$-**self-affine** $\mathcal{T}$.

prototile

**subdivision:**

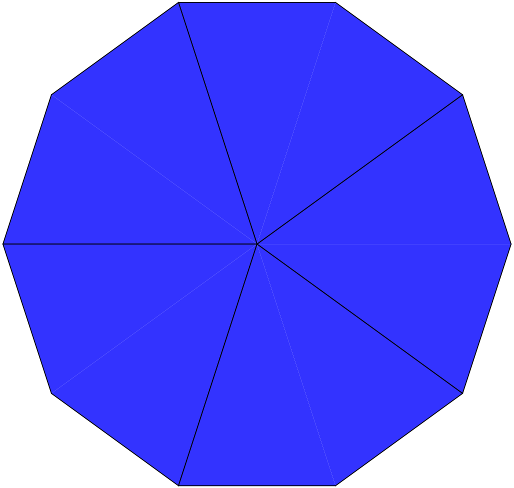
(to be done twice)



**inflation:**

$$\phi = \begin{bmatrix} \lambda^2 & 0 \\ 0 & \lambda^2 \end{bmatrix} \quad \lambda = \frac{\sqrt{5}+1}{2}$$

# "Sun" Kite-and-Dart Tiling Generation



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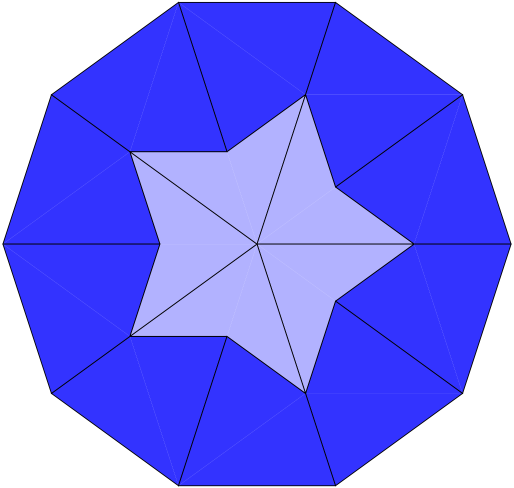

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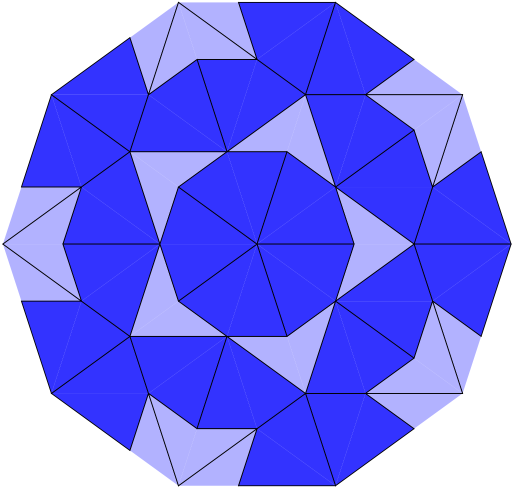

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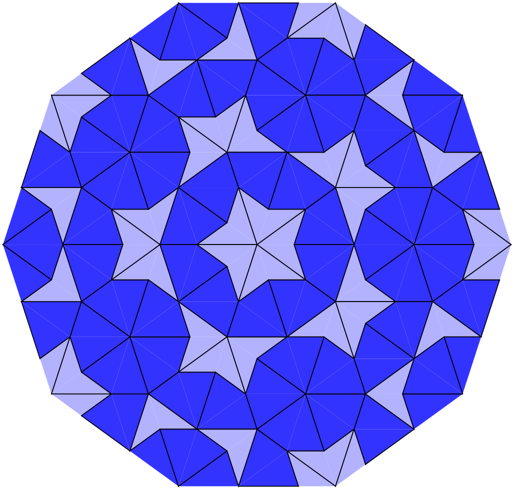

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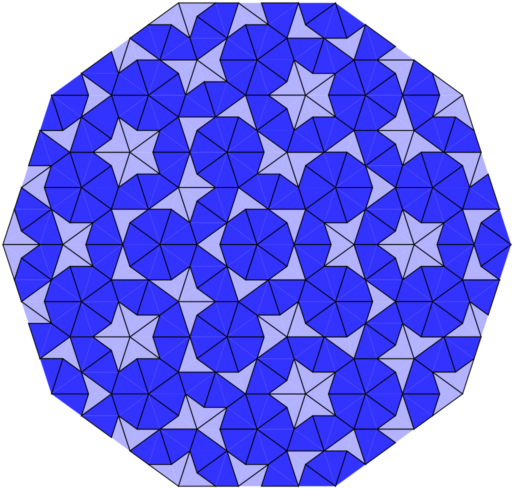

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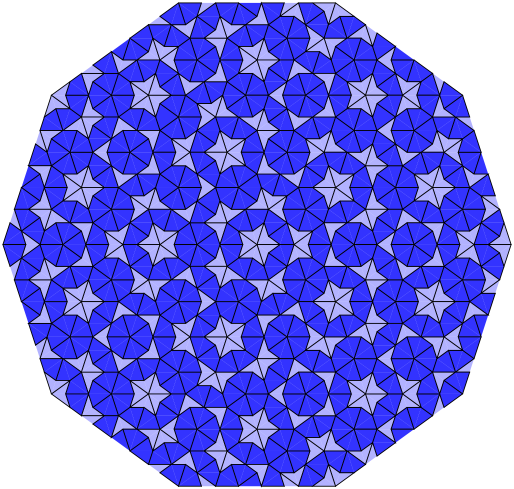
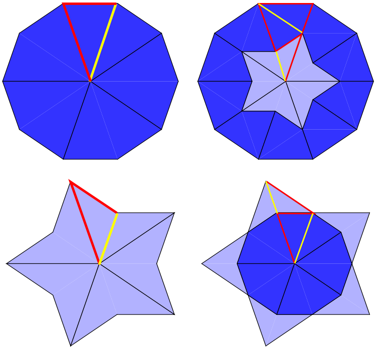


# Primitivity Hypothesis

We assume that the **subdivision matrix** $A = (a_{ij})$ is **primitive**, or even $A > 0$ (by passing to an iterate).



$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^2$$

# Outline

# Grand Challenge

Describe **all** self-affine tilings.

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OK, start with all the expansion factors $\phi$.

# Thurston-Kenyon-Solomyak Theorem

## Theorem

*If $\phi$ is a diagonalizable expansion factor then*
*(i) $\phi$ is* **integral algebraic**,
*(ii) $\phi$ is* **Perron**.

where, for *diagonalizable* $\phi$:

**integral algebraic** $\equiv \lambda \in \text{spec}(\phi) \implies \lambda$ is an algebraic integer

**Perron** $\equiv \begin{pmatrix} \lambda, \gamma \text{ alg. conjugate} \\ |\gamma| \geq |\lambda| \end{pmatrix} \implies \text{mult}_\phi(\gamma) \geq \text{mult}_\phi(\lambda)$

i.e.

$$\text{spec}(\phi) = \begin{array}{l} \boxed{\lambda_1\ \lambda_2\ \lambda_3\ \lambda_4}\ \lambda_5\ \lambda_6\ \ldots \\ \boxed{\lambda_1\ \lambda_2\ \lambda_3\ \lambda_4}\ \lambda_5\ \lambda_6\ \ldots \\ \boxed{\lambda_1\ \lambda_2\ \lambda_3}\ \lambda_4\ \lambda_5\ \lambda_6\ \ldots \\ \boxed{\lambda_1\ \lambda_2}\ \lambda_3\ \lambda_4\ \lambda_5\ \lambda_6\ \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

(alg conj classes in rows)

# Thurston-Kenyon-Solomyak Theorem +

## Theorem

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where,

**integral algebraic** $\equiv \lambda \in \text{spec}(\phi) \implies \lambda$ is an algebraic integer

$$\textbf{Perron} \equiv \begin{pmatrix} \lambda, \gamma \text{ alg. conj.} \\ |\gamma| \geq |\lambda|, \ r \in \mathbb{N} \end{pmatrix} \implies \text{mult}_\phi(\gamma, r) \geq \text{mult}_\phi(\lambda, r)$$

($\text{mult}_\phi(\lambda, r)$ is the multiplicity of $(\lambda, r)$ in **Jordan spectrum of $\phi$**.)

# Jordan spectrum of $\phi$

is a **list**

$$\text{Jspec}(\phi) := \{(\lambda_1, r_1), (\lambda_2, r_2), \ldots, (\lambda_k, r_k)\}$$

with $\lambda_i \in \mathbb{C}/(\lambda \sim \overline{\lambda})$, $r_i \in \mathbb{N}$, and such that

$$\textbf{Jordan form of } \phi = J_{\lambda_1, r_1} \oplus J_{\lambda_2, r_2} \oplus \ldots \oplus J_{\lambda_k, r_k}$$

where the **Jordan block** is

$$J_{\lambda,r} = \begin{bmatrix} \Lambda_\lambda & I_\lambda & & \\ & \Lambda_\lambda & I_\lambda & \\ & & \ldots & \\ & & & \Lambda_\lambda \end{bmatrix} \quad (r \times r) \tag{1}$$

with

$$\Lambda_\lambda = \lambda \ \text{ or } \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \quad \text{and} \quad I_\lambda = 1 \ \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

depending on

$$\lambda \in \mathbb{R} \ \text{ or } \ \lambda = \alpha + i\beta \text{ with } \beta \neq 0.$$

# Simple Examples

**Perron** $\phi$:

$$\phi_{\text{good}} = \begin{bmatrix} 3+\sqrt{2} & 1 & 0 & 0 \\ 0 & 3+\sqrt{2} & 0 & 0 \\ 0 & 0 & 3+\sqrt{2} & 0 \\ 0 & 0 & 0 & 3-\sqrt{2} \end{bmatrix}$$

$$\text{Jspec}(\phi_{\text{good}}) = \{(3+\sqrt{2},2),(3+\sqrt{2},1),(3-\sqrt{2},1)\}$$

**non-Perron** $\phi$:

$$\phi_{\text{bad}} = \begin{bmatrix} 3+\sqrt{2} & 1 & 0 \\ 0 & 3+\sqrt{2} & 0 \\ 0 & 0 & 3-\sqrt{2} \end{bmatrix}$$

$$\text{Jspec}(\phi_{\text{bad}}) = \{(3+\sqrt{2},2),\ \cancel{(3+\sqrt{2},1)},(3-\sqrt{2},1)\}$$

*Corollary: As conjectured by K-S,* $\phi_{bad}$ *is not an expansion factor.*

# Novelty?

We reuse or adapt 64.8% of the K-S argument and add new 35.2%, including swapping the **heart**:

| | | |
|---|---|---|
| **Rademacher's a.e. differentiability** of a Lipschitz function | $\longrightarrow$ | **ergodic averaging** of a **novel cocycle** |

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This neutralizes the chief annoyance of non-diagonalizable $\phi$: **invariant subspaces need not be invariantly complemented**.

# Outline

# Trade the Tiling for a **Substitution Delone Multiset**

In each prototile $T_i$, $i = 1, \ldots, \kappa$, fix **Thurston's control point**:

$$\text{tiling } \mathcal{T} \quad \rightsquigarrow \quad \text{Delone multiset } \boldsymbol{\Lambda} = (\Lambda_i)_{i=1}^{\kappa}$$

where

$$\phi \Lambda_i \subset \Lambda_i$$

Inflation-substitution invariance of $\mathcal{T}$ translates to

$$\Lambda_i = \text{ union of finitely many translates of } \phi \Lambda_j \quad (j = 1, \ldots, \kappa)$$

Specifically, there are finite $D_{ij} \subset \mathbb{R}^d$ so that

$$\Lambda_i = \bigcup_{j=1}^{\kappa} \phi \Lambda_j + D_{ij} \quad (i = 1, \ldots, \kappa),$$

denoted

$$\boldsymbol{\Lambda} = \Phi(\boldsymbol{\Lambda}).$$

# The **Lattice** and the **Mathematical Space**

$\mathbb{Z}$**-module** $\mathcal{J} \subset \mathbb{R}^d$ generated by the control points

$$\mathcal{J} := \left\{ \sum_k a_k v_k : \; a_k \in \mathbb{Z}, \; v_k \in \bigcup_{i=1}^{\kappa} \Lambda_i \right\}$$

is of rank $N := \mathrm{rank}(\mathcal{J}) < \infty$ so $1 \otimes_{\mathbb{Z}} \mathcal{J}$ is a **lattice** in

$$\mathcal{V} := \mathbb{R} \otimes_{\mathbb{Z}} \mathcal{J} = \left\{ \sum_k a_k (1 \otimes_{\mathbb{Z}} v_k) : \; a_k \in \mathbb{R} \right\},$$

an $N$-dim linear space over $\mathbb{R}$ with the **projection** $V : \mathcal{V} \to \mathbb{R}^d$

$$V : 1 \otimes_{\mathbb{Z}} v \mapsto v.$$

$\phi$ **lifts** to a linear $M : \mathcal{V} \to \mathcal{V}$

$$M(1 \otimes_{\mathbb{Z}} v) := 1 \otimes_{\mathbb{Z}} (\phi v).$$

$M$ is **integral** (preserves the lattice) and **factors** onto $\phi$:

$$V \circ M = \phi \circ V$$

so $\phi$ is integral algebraic. *50% done? Not quite!*

# Basic example: Fibonacci

$$\phi(x) = \lambda \quad \text{where } \lambda = \frac{\sqrt{5}+1}{2}.$$

$$\mathcal{J} = \langle 1, \sqrt{5} \rangle = \mathbb{Z}[\lambda].$$

$$\mathcal{V} \simeq \mathbb{R}^2 \text{ and } 1 \otimes_{\mathbb{Z}} \mathcal{J} \simeq \mathbb{Z}^2.$$

$$M \simeq \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

# Quo vadis?

PLAN of ACTION:

**Lift** all structures
from the **physical space** $\mathbb{R}^d$
to the **mathematical space** $\mathcal{V}$
(where life is better!)

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To succeed, pack more stuff than T-K-S did.

# To be lifted: **Tiling Space** ($\equiv$ **Hull** of $\boldsymbol{\Lambda}$)

$$\mathcal{X} := \mathrm{cl}\{\boldsymbol{\Lambda} - t : \ t \in \mathbb{R}^d\}$$

The **canonical lifting** takes $\boldsymbol{\Lambda}$ to a multiset in $\tilde{\boldsymbol{\Lambda}}$ in $\mathcal{V}$,

$$\tilde{\boldsymbol{\Lambda}} := \{i@(1 \otimes_{\mathbb{Z}} p) : \ i@p \in \boldsymbol{\Lambda}\}.$$

For $\mathbf{x} = \lim_{k\to\infty} \boldsymbol{\Lambda} - t_k \in \mathcal{X}$, take $\tilde{\mathbf{x}} := \lim_{k\to\infty} \tilde{\boldsymbol{\Lambda}} - \tilde{t}_k$ where $\tilde{t}_k \in V^{-1}(t_k)$ are selected at will (ensuring convergence).

$\tilde{\mathbf{x}}$ is a multiset in $\mathcal{V}$ with $V(\tilde{\mathbf{x}}) = \mathbf{x}$ and has **lattice difference**:

$$\tilde{\mathbf{x}} - \tilde{\mathbf{x}} \subset 1 \otimes_{\mathbb{Z}} \mathcal{J}.$$

The **lift** $\tilde{\mathbf{x}}$ is well defined only modulo translation along $\mathcal{K}$ where

$$\mathcal{K} := \ker V.$$

Q: Can we "peel away" the **mod** $\mathcal{K}$ ambiguity in $\tilde{\mathbf{x}}$?

# Step 1: Lift the **Inflation-Substitution Map** $\Phi$

The **inflation-substitution map** $\Phi$ (fixing $\boldsymbol{\Lambda}$) acts on colored points by

$$\Phi(j@p) = \{i@(\phi p + d) : d \in D_{ij}\}.$$

One can lift $D_{ij} \subset \mathbb{R}^d$ to $\tilde{D}_{ij} \subset \mathcal{V}$ so that $\tilde{D}_{ij} - \tilde{D}_{ij} \subset 1 \otimes_{\mathbb{Z}} \mathcal{J}$.

Thus $\Phi$ **lifts** to $\tilde{\Phi}$ acting on multisets in $\mathcal{V}$ and given on colored points by

$$\tilde{\Phi}(j@\tilde{p}) = \{i@(M\tilde{p} + \tilde{d}) : \ \tilde{d} \in \tilde{D}_{ij}\}.$$

By construction,

$$V \circ \tilde{\Phi} = \Phi \circ V$$

*NOTE: $\tilde{\Phi}$ is unique up to the harmless $\tilde{\Phi} \rightsquigarrow \tilde{\Phi} + v$ where $v \in \mathcal{K}$.*

*(In the write-up, I needlessly normalize this away by arranging $D_{ij} \subset \mathcal{J}$.)*

## Step 2: Lift Φ-**orbits** (still only mod $\mathcal{K}$)

$M$ has **stable**, **unstable**, **central**, **and eventual kernel** subspaces:

$$\mathcal{V}^s \oplus \mathcal{V}^u \oplus \mathcal{V}^c \oplus \mathcal{V}^0 = \mathcal{V}$$

(Since $M/\mathcal{K} \simeq \phi$ and $\phi$ is expanding $\mathcal{V}^s \oplus \mathcal{V}^c \oplus \mathcal{V}^0 \subset \mathcal{K}$.)

Given $\mathbf{x} \in \mathcal{X}$, lifting each $\Phi^n(\mathbf{x})$ in the Φ-**orbit** $(\Phi^n(\mathbf{x}))_{n=0}^{\infty}$ gives a sequence $(\tilde{\mathbf{x}}_n)_{n=0}^{\infty}$ with

$$\tilde{\Phi}(\tilde{\mathbf{x}}_n) = \tilde{\mathbf{x}}_{n+1} + t_{n+1} \quad \text{where } t_{n+1} \in \mathcal{K} \quad (n \geq 0).$$

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An easy adjustment*, $\tilde{\mathbf{x}}_n^{\text{new}} := \tilde{\mathbf{x}}_n + \tau_n$, secures $t_n^u = 0$, so that $(\tilde{\mathbf{x}}_n)_{n=0}^{\infty}$ is an $\tilde{\Phi}_u$-**orbit**, i.e.,

$$\tilde{\Phi}(\tilde{\mathbf{x}}_n)^u = \tilde{\mathbf{x}}_{n+1}^u \quad (n \geq 0)$$

*∗ this is obvious, unless $n \in \mathbb{Z}$ and $\tau_n^{su} := -(M_{su} - I)^{-1} t_{n+1}^{su}$, using hyperbolicity of $M_{su} := M|_{\mathcal{V}^{su}}$.*

# Step 3: **Unstably Well Position** the $\tilde{\Phi}_u$-Orbits

The basic theory of the hyperbolic non-stationary recurrence

$$t_{n+1} = M_u t_n + d_n \quad \text{with bounded } d_n \quad (n \geq 0, t_n, d_n \in \mathcal{V}^u)$$

produces a **finer adjustment** $\tilde{\mathbf{x}}_0^{\text{new}} := \tilde{\mathbf{x}}_0 + \tau$ turning $(\tilde{\mathbf{x}}_n)_0^\infty$ into **(unstably) non-escaping sequence** of multisets, i.e.,

$$\mathrm{dist}\,(0^u, \tilde{\mathbf{x}}_n^u) := \inf\{|p^u| : \ i@p \in \tilde{\mathbf{x}}_n\} < C \quad (\exists_{C>0}\ \forall_{n\geq 0}).$$

A lift $\tilde{\mathbf{x}}$ of $\mathbf{x} \in \mathcal{X}$ is **well positioned** iff $\tilde{\mathbf{x}} = \tilde{\mathbf{x}}_0$ where $(\tilde{\mathbf{x}}_n)$ is as above. The unstable component $\tilde{\mathbf{x}}^u$ is determined uniquely!

Finally, the **(well) lifted tiling space** is

$$\tilde{\mathcal{X}} := \{\tilde{\mathbf{x}} : \ \tilde{\mathbf{x}} \text{ is a well positioned lift of } \mathbf{x} \in \mathcal{X}\}$$

*NOTE: $\tilde{\mathbf{x}} \in \tilde{\mathcal{X}}$ is a multiset in $\mathcal{V}$ mod $\mathcal{V}^{sc0} := \mathcal{V}^s \oplus \mathcal{V}^c \oplus \mathcal{V}^0$.*

(Could do mod $\mathcal{V}^{c0} := \mathcal{V}^c \oplus \mathcal{V}^0$ by using $n \in \mathbb{Z}$.)

# Where are we now?

$$\mathbb{R}^d \xleftarrow{\text{PROJECTION } V} \mathcal{V}^{\color{red}u}$$

$$\phi \xleftarrow{\text{FACTORING BY } V} M^{\color{red}u}$$

$$\Phi \xleftarrow{\text{FACTORING BY } V} \tilde{\Phi}$$

$$\mathcal{X} = \Phi(\mathcal{X}) \xrightarrow[\color{blue}\text{modulo } \mathcal{V}^{sc0}]{\text{WELL POSITIONED LIFTING}} \tilde{\mathcal{X}} = \tilde{\Phi}(\tilde{\mathcal{X}})$$

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MISSING: the translation induced **minimal** $\mathbb{R}^d$**-action** on $\mathcal{X}$.

# The **Cocycle** $\alpha^u$

$$\mathbf{x} + t \xrightarrow{\text{WELL POSITIONED LIFTING}} \tilde{\mathbf{x}} \; + \; ???$$

Given $\mathbf{x} \in \mathcal{X}$ and $t \in \mathbb{R}^d$, let $\alpha^u(\mathbf{x}, t) \in \mathcal{V}^u$ be such that

$$(\mathbf{x} + t)\tilde{} = \tilde{\mathbf{x}} + \alpha^u(\mathbf{x}, t) \qquad \pmod{\mathcal{V}^{sc0}}.$$

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Note that $V \circ \alpha^u(\mathbf{x}, t) = t$ and $\alpha^u$ is a **cocycle**

$$\alpha^u(\mathbf{x}, t + s) = \alpha^u(\mathbf{x}, t) + \alpha^u(\mathbf{x} + t, s).$$

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Fact 1: $\alpha^u(\mathbf{x}, t)$ is Hölder in $t$ because it is **self-affine**:

$$M_u \circ \alpha^u(\mathbf{x}, t) = \alpha^u(\Phi\mathbf{x}, \phi t).$$

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Fact 2: $\alpha^u(\mathbf{x}, t)$ is transversally locally constant in $\mathbf{x} \in \mathcal{X}$:

$$\mathbf{x}|_R = \mathbf{y}|_R \text{ and } |s| < R \implies \alpha^u(\mathbf{y}, s) = \alpha^u(\mathbf{x}, s).$$

# The **Cocycle** $\alpha^u$

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Fact 3: $\alpha^u(\mathbf{x}, t)$ is given by canonical lifting on **return vectors**:

$$\mathbf{x}+t|_R = \mathbf{x}|_R \implies t \in \mathcal{J} \text{ and } \alpha^u(\mathbf{x}, t) = (1 \otimes_{\mathbb{Z}} t)^u.$$

# **Ergodic Averaging** of the cocycle $\alpha^u$

The translation action on $\mathcal{X}$ is **uniquely ergodic** so there is a linear transformation $A^u : \mathbb{R}^d \to \mathcal{V}^u$ such that

$$\alpha^u(\mathbf{x}, t) = A^u t + \mathrm{Error}(\mathbf{x}, t) \quad \text{with} \quad |\mathrm{Error}(\mathbf{x}, t)| = o(|t|).$$

The approx is uniform: $\lim_{R\to 0} \sup_{\mathbf{x}\in\mathcal{X}, |t|=R} |\mathrm{Error}(\mathbf{x}, t)|/|t| = 0$.

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From $V \circ \alpha^u(\mathbf{x}, t) = t$, we get $V \circ A = Id_{\mathbb{R}^d}$ so

$$A^u : \mathbb{R}^d \xrightarrow{\sim} \mathcal{E} := \{At : \ t \in \mathbb{R}^d\} \subset \mathcal{V}^u.$$

From $M^u \circ \alpha^u(\mathbf{x}, t) = \alpha^u(\Phi\mathbf{x}, \phi t)$, we get $M \circ A^u = A^u \circ \phi$ so our old friend $\mathcal{K} = \ker(V)$ is **invariantly complemented**:

$$\mathcal{V} = \mathcal{E} \oplus \mathcal{K} \quad \text{and} \quad M|_{\mathcal{E}} \simeq \phi$$

... which is better than our previous $\phi \simeq M/\mathcal{K}$.

# RECAP

Level 1:
$\phi : \mathbb{R}^d \to \mathbb{R}^d$, tiling $\mathcal{T}$ ($\equiv$ multiset $\mathbf{\Lambda}$), inflation-substitution rule $\Phi$.

Level 2:
$\phi : \mathbb{R}^d \to \mathbb{R}^d$, tiling space $\mathcal{X}$, $\Phi : \mathcal{X} \to \mathcal{X}$, translation $\mathbb{R}^d$-action.

Level 3:
$M : \mathcal{V} \xrightarrow{int.} \mathcal{V}$, lifted tiling space $\tilde{\mathcal{X}}$, $\tilde{\Phi} : \tilde{\mathcal{X}} \to \tilde{\mathcal{X}}$, cocycle $\alpha^u(\mathbf{x}, t)$.

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Keep in mind

$$\mathcal{V} = \mathcal{E} \oplus \mathcal{K}, \quad M|_{\mathcal{E}} \simeq \phi : \mathbb{R}^d \to \mathbb{R}^d$$

and the corresponding actions are conjugate:

$$\tilde{\mathcal{X}} \simeq \mathcal{X}, \quad \tilde{\Phi} \simeq \Phi, \quad (\mathbf{x} \mapsto \mathbf{x} + t) \simeq (\tilde{\mathbf{x}} \mapsto \tilde{\mathbf{x}} + \alpha^u(\mathbf{x}, t)) \, .$$

# Outline

# Line of Attack

*Have to prove the other 50%:*

$$\textbf{Perron} \equiv \begin{pmatrix} \lambda, \gamma \text{ alg. conj.} \\ |\gamma| \geq |\lambda|, \; r \in \mathbb{N} \end{pmatrix} \implies \mathrm{mult}_\phi(\gamma, r) \geq \mathrm{mult}_\phi(\lambda, r)$$

Assume then

$$|\gamma| \geq |\lambda| > 1 \quad \text{and} \quad \gamma \notin \{\lambda, \overline{\lambda}\}.$$

Take the **real eigenspace** of $\phi$ with eigenvalue $\lambda$:

$$E_\lambda \overset{*}{=} \bigoplus \{\text{Jordan space with eigenvalue } \lambda\} \subset \mathbb{R}^d$$

Take the **real eigenspace** of $M$ with eigenvalue $\gamma$:

$$\mathcal{V}_\gamma = \bigoplus \{\text{Jordan space with eigenvalue } \gamma\} \subset \mathcal{V}^u.$$

| |
|---|
| GOAL: Show that $\alpha_\gamma(\mathbf{x}, \cdot)$ linearly(!) factors $\phi$ **onto** $M|_{\mathcal{V}_\gamma}$. |

(As then $\phi$ has as many $(\gamma, r)$-Jordan blocks as $M|_{\mathcal{V}_\gamma}$ … DONE!)

* $E_\lambda$ is indep. of the Jordan decomp.; $E_\lambda = \ker(\phi - \lambda I)^\infty$, $\lambda \in \mathbb{R}$.

# Cocycle $\alpha^u$ is **Lyapunov Exponent Shy**:

Fact 4: $|\gamma| \geq |\lambda| \implies$ $\gamma$-component of $\alpha^u$ vanishes on $E_\lambda$:

$$\alpha_\gamma(\mathbf{x}, \cdot)|_{E_\lambda} = 0.$$

(We use invariant splittings $\mathbb{R}^d = E_\lambda \oplus \ldots$ and $\mathcal{V} = \mathcal{V}_\gamma \oplus \ldots$.)

*Idea of Proof:* Suppose that $\alpha_\gamma(\mathbf{x}, v) \neq 0$ for $v \in E_\lambda$. Then

$$\frac{|A_\gamma \phi^n v|}{|\phi^n v|} \approx \frac{|\alpha_\gamma(\Phi^n \mathbf{x}, \phi^n v)|}{|\phi^n v|} = \frac{|M^n \alpha_\gamma(\mathbf{x}, v)|}{|\phi^n v|} \overset{*}{\geq} \frac{C^{-1}|\gamma|^n}{C|\lambda|^n} \geq C^{-2} > 0.$$

Thus $A_\gamma|_{E_\lambda} \neq 0$; so $A_\gamma|_{E_\lambda}$ **non-trivially factors** $\phi|_{E_\lambda}$ into $M|_{\mathcal{V}_\gamma}$, contradicting $\gamma \notin \{\lambda, \overline{\lambda}\}$. □

$*$ **cheated!** in non-diag case when $|\gamma| = |\lambda|$ and $|\phi^n v| \sim t^r |\lambda|^n$.

*(K-S argue at small scale $\approx 0$, JK at large scale $\approx \infty$.)*

Fact 5: $\alpha_\gamma(\mathbf{x}, \cdot)$ is **linear*** on all of $\mathbb{R}^d$

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*Idea of Proof (K-S):* Need **return vectors**,

$$\Theta := \{p - q : \; i@p, i@q \in \mathbf{\Lambda}, \; i = 1, \ldots, m\}.$$

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If $t \in \Theta$ and $s$ is tiny then **local transverse constancy** (F2) gives:

$$\alpha_\gamma(\mathbf{x}, t + s) = \alpha_\gamma(\mathbf{x}, t) + \alpha_\gamma(\mathbf{x} + t, s) = \alpha_\gamma(\mathbf{x}, t) + \alpha_\gamma(\mathbf{x}, s)$$

Roughly,

*"$\alpha_\gamma(\mathbf{x}, \cdot)$ is linear on tiny return vectors."*

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Roughly,

*"$\alpha_\gamma(\mathbf{x}, \cdot)$ is linear on tiny return vectors."*

*"tiny return vectors"* is an oxymoron! But any $t \in \mathbb{R}^d$ decomposes into $Pt$ in $E_\lambda$ and $Qt$ in the **complementary invariant space**.

**Shyness** (F4), $\alpha_\gamma(\mathbf{x}, \cdot)|_{E_\lambda} = 0$, gives

$$\alpha_\gamma(\mathbf{x}, t) = \alpha_\gamma(\mathbf{x}, Pt+Qt) = \alpha_\gamma(\mathbf{x}, Qt) + \alpha_\gamma(\mathbf{x}+Qt, Pt) = \alpha_\gamma(\mathbf{x}, Qt).$$

If only the return vectors $t \in \Theta$ with tiny $Qt$ spanned* $\mathbb{R}^d$, linearity would follow. □

*cheated!: *Adaptation of ceratin "K-S gymnastics" needed here!*

## Fact 6: $\alpha_\gamma(\mathbf{x}, \cdot) : E_\lambda \to \mathcal{V}_\gamma$ is **onto** (and the Grand Finale!)

*Proof:*

By "$t$ **ret. vect.** $t \implies \alpha^u(\mathbf{x}, t) =$ **the canonical lift of** $t$" (F3):

$$\begin{aligned} \alpha_\gamma(\mathbf{x}, \mathbb{R}^d) &= \operatorname{span}\{\alpha_\gamma(\mathbf{x}, v) : \ v \in \Theta\} \\ &= \operatorname{span}\{\pi_\gamma(1 \otimes_\mathbb{Z} v)^u : \ v \in \Theta\} = \mathcal{V}_\gamma. \end{aligned}$$

(We used that $\operatorname{span}(\Theta) = \mathbb{R}^d$ and $\operatorname{span}(1 \otimes_\mathbb{Z} \Theta) = \mathcal{V}$.) □

Got "$\alpha_\gamma(\mathbf{x}, \cdot)$ linearly factors $\phi$ **onto** $M|_{\mathcal{V}_\gamma}$"

and can get the desired $\mathrm{mult}_\phi(\gamma, r) \geq \mathrm{mult}_\phi(\lambda, r)$ as promised:

$$\begin{aligned} \mathrm{mult}_\phi(\gamma, r) &\geq \mathrm{mult}_{M|_{\mathcal{V}_\gamma}}(\gamma, r) && \text{(by factoring)} \\ &= \mathrm{mult}_M(\gamma, r) && \text{(by def of } \mathcal{V}_\gamma\text{)} \\ &= \mathrm{mult}_M(\lambda, r) && \text{(since } \gamma, \lambda \text{ alg. conj.)} \\ &\geq \mathrm{mult}_\phi(\lambda, r) && \text{(since } \phi \simeq M|_\mathcal{E}\text{).} \end{aligned}$$

*Thank you!*

*(Could have taken $\mathbf{x} = \Lambda$.)*

# Outline

# Pisot and Meyer Property

**Theorem (Lee-Solomyak+)**

*For any* $\phi$-self-affine tiling:*
*$\phi$ is Pisot $\implies$ $\boldsymbol{\Lambda}$ has Meyer property (i.e. $\boldsymbol{\Lambda} - \boldsymbol{\Lambda}$ is uniformly discrete).*

# Pisot and Meyer Property

**Theorem (Lee-Solomyak+)**

*For any* $\phi$-self-affine tiling:*
*$\phi$ is Pisot $\implies$ $\mathbf{\Lambda}$ has Meyer property (i.e. $\mathbf{\Lambda} - \mathbf{\Lambda}$ is uniformly discrete).*

*Idea of Proof:*
Pisot property $\equiv \mathcal{V}^u = \mathcal{E}$.
Well postioning $\implies \exists_{R>0} \forall_{\tilde{\mathbf{x}} \in \tilde{\mathcal{X}}} \tilde{\mathbf{\Lambda}} \subset B_R(\mathcal{E})$.
Hence, $\tilde{\mathbf{\Lambda}} - \tilde{\mathbf{\Lambda}} \subset B_R(\mathcal{E}) - B_R(\mathcal{E}) = B_{2R}(\mathcal{E})$.
So $\mathbf{\Lambda} - \mathbf{\Lambda} = V(\tilde{\mathbf{\Lambda}} - \tilde{\mathbf{\Lambda}}) \subset V(B_{2R}(\mathcal{E}))$.
Finally, $V(B_{2R}(\mathcal{E}))$ is tautologically uniformly discrete. □
$*$: Need to take an inverse limit or assume aperiodicity so that $\Phi$ is a homeo.