Sturmian colorings of regular trees

(joint work with Seonhee Lim)

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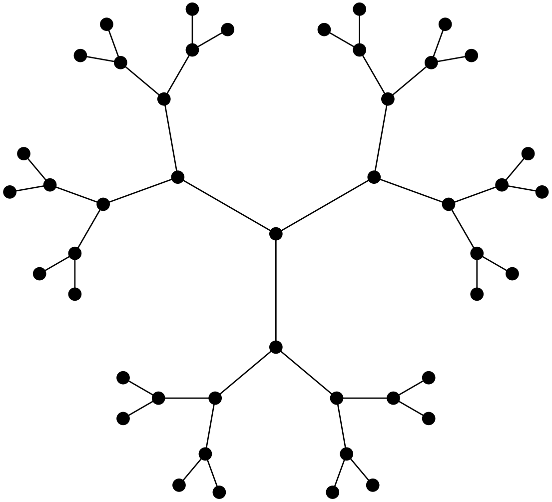
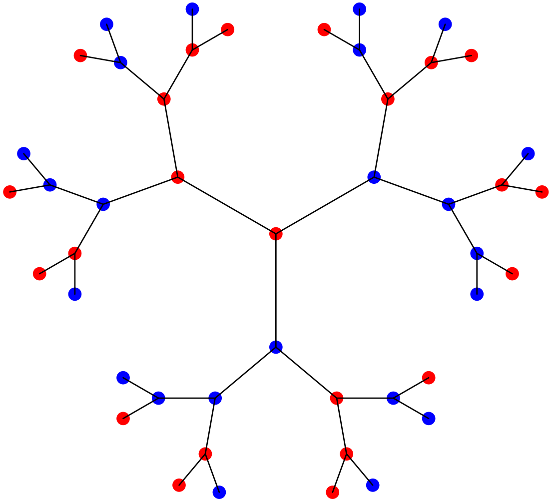
A regular tree (3-regular tree)



T is k-regular tree, i.e., # of edges with an initial vertex is k.

All edge lengths = 1.

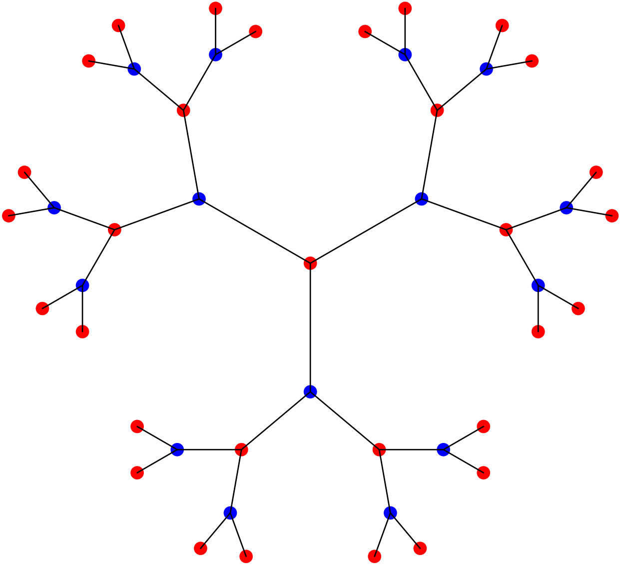
A vertex coloring of 3-regular tree



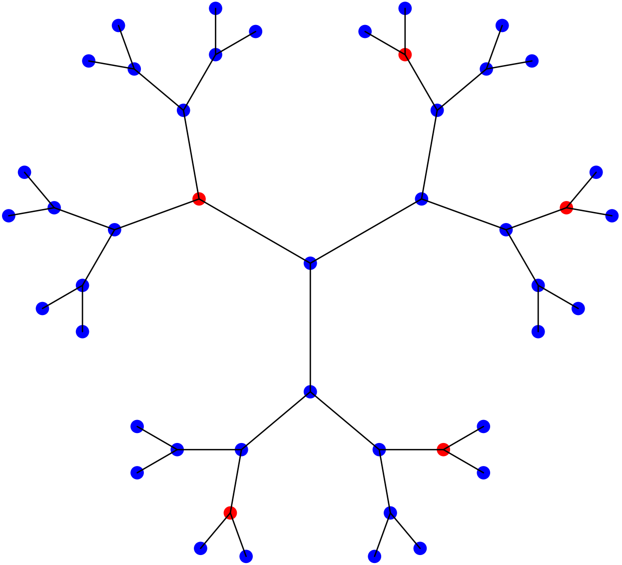
A coloring of T is a map $\phi : VT \to \mathcal{A}$.

VT : vertex set of T.
$\mathcal{A}$: alphabet.

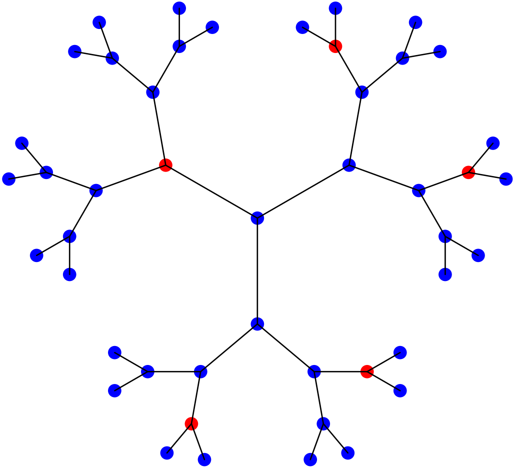
A periodic coloring of 3-regular tree

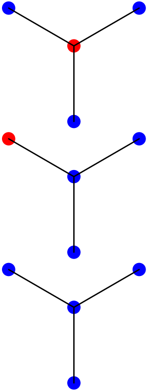


Another periodic coloring of 3-regular tree

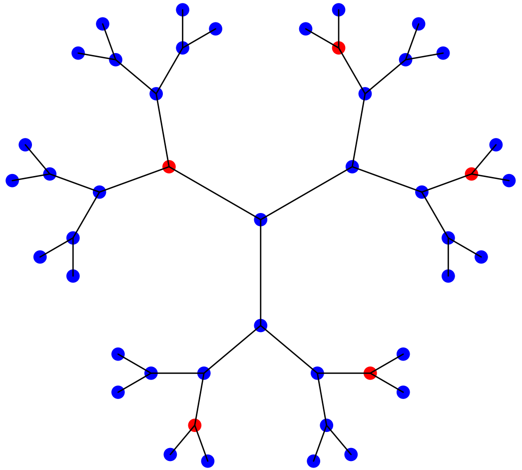


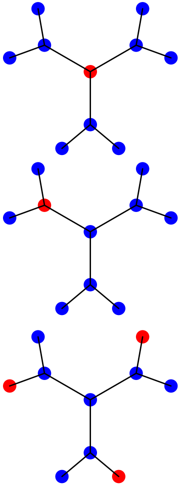
$B_n(x) = \{y \in VT \cup ET : d(x, y) \leq n\}$. ($ET$: edge set of T)
$[B_n(x)]$: equivalent class by color-preserving isomorphisms.





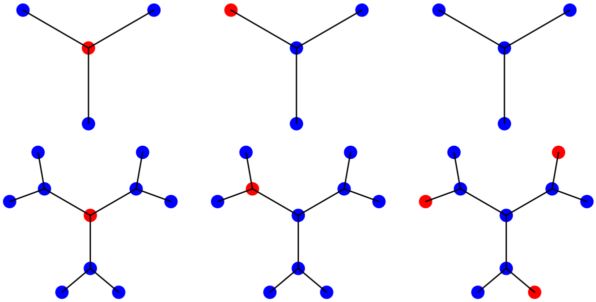
1-balls





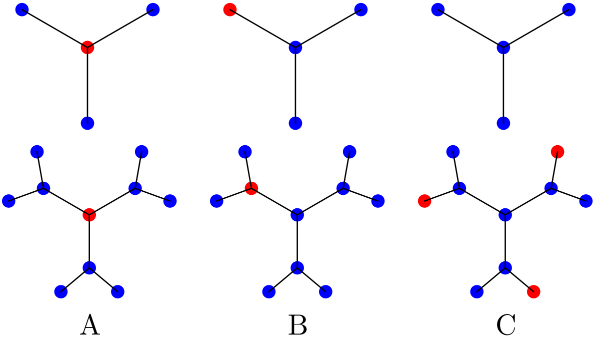
2-balls

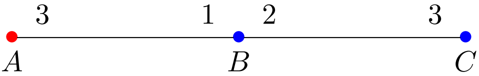
Finite quotient graph



A B C

Finite quotient graph





Periodic coloring on the regular tree

$G = \text{Aut}(T)$: the group of automorphisms of T, a locally compact topological group with compact-open topology.

Definition

A coloring $\phi : VT \to \mathcal{A}$ is periodic if there exists a subgroup $\Gamma \subset G$ such that $\Gamma \backslash T$ is a finite graph and ϕ is Γ-invariant, i.e.

$$\phi(\gamma x) = \phi(x), \text{ for all } x \in VT \text{ and } \gamma \in \Gamma.$$

Note that we do not require Γ to be a discrete subgroup of G.

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Theorem

Let $\phi : VT \to \mathcal{A}$ be a coloring. The followings are equivalent.

1. *The coloring ϕ is periodic.*
2. *The subword complexity of ϕ satisfies $b_\phi(n+1) = b_\phi(n)$ for some $n > 0$.*
3. *The subword complexity $b_\phi(n)$ is bounded.*

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3. *The subword complexity $b_\phi(n)$ is bounded.*

Definition

A coloring ϕ of a k-regular tree T is called Sturmian if $b_\phi(n) = n + 2$.

Infinite word

The subword complexity for an infinite word u

$$p_u(n) = \text{ the number of different subwords of length } n \text{ in } u.$$

Theorem (Hedlund-Morse)

$p_u(n)$ is bounded if and only if u is eventually periodic.

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Note that in the 2-regular tree, colored n-balls are $2n + 1$-words indentified with revered words.

Example

Let $\Gamma = \langle a_1, \cdots, a_k : a_i^2 = 1 \rangle$ and T be the Cayley graph,

$$g \in \mathrm{Aut}(T)$$

For $t \in VT$, a unique $\gamma_t \in \Gamma$ sending the identity to t. Then

$$\gamma_{g(t)}^{-1} \circ g \circ \gamma_t(\mathrm{id}) = \mathrm{id}.$$

Let $\phi_g(t)$ be the map $\gamma_{g(t)}^{-1} \circ g \circ \gamma_t$ restricted to the 1-sphere of the identity. Then

$$\phi_g : VT \to S_k \text{ is a coloring,}$$

where S_k is the symmetric group.

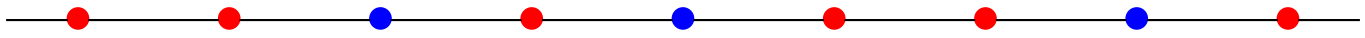
Lubotzky, Mozes and Zimmer showed that

$$\phi_g \text{ is periodic } \Leftrightarrow \ g \in \mathrm{Comm}(\Gamma).$$

Colorings on the 2-regular tree and bi-infinite words

A bi-infinite Sturmian word:

$$\cdots \; a \; b \; \underline{a \; a \; b \; a \; b \; a \; a \; b \; a} \; a \; b \; \cdots$$



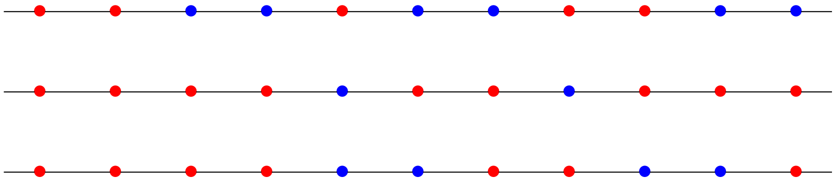
Colorings on the 2-regular tree and bi-infinite words

A bi-infinite Sturmian word:

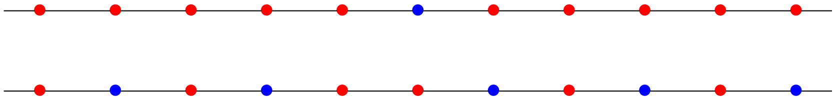
$$\cdots \ a \ b \ \underline{a \ a \ b \ a \ b \ a \ a \ b \ a} \ a \ b \ \cdots$$

$$\begin{aligned} b(n) &= \frac{1}{2} \cdot \Big(p(2n+1) + \ \# \text{ of palindromic } (2n+1)\text{-word} \Big) \\ &= \frac{1}{2} \cdot \Big((2n+2) + 2 \Big) = n + 2. \end{aligned}$$

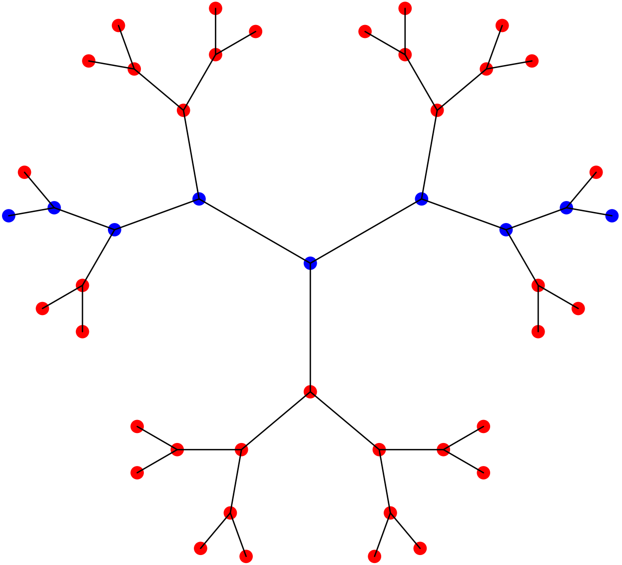
Other Sturmian colorings on the 2-regular tree

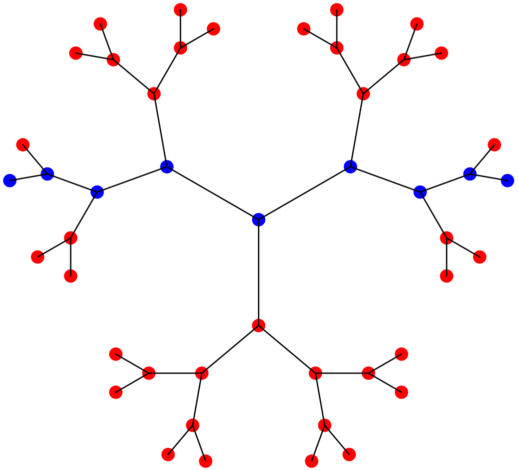


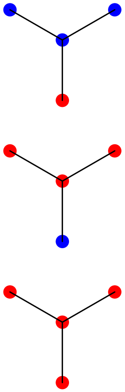
Also “non-irrational ” colorings:



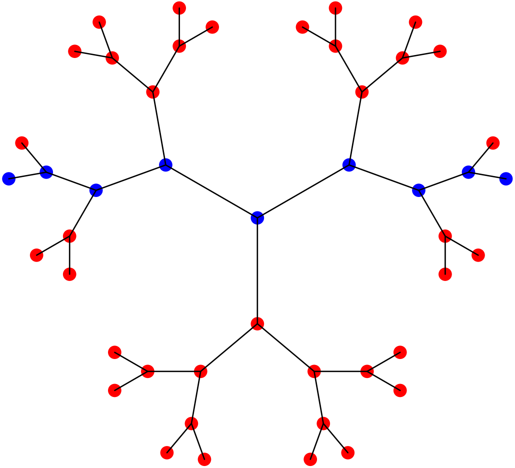
A sturmian coloring of 3-regular tree

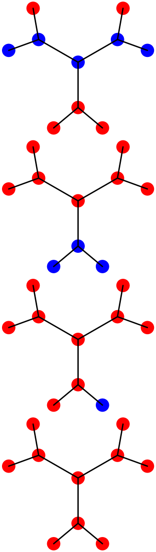




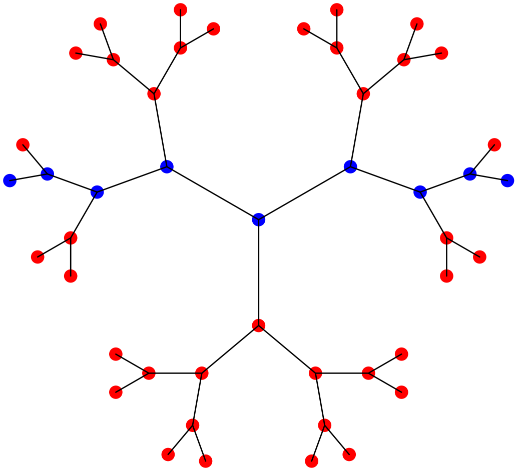


$$b(1) = 3$$

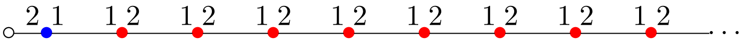




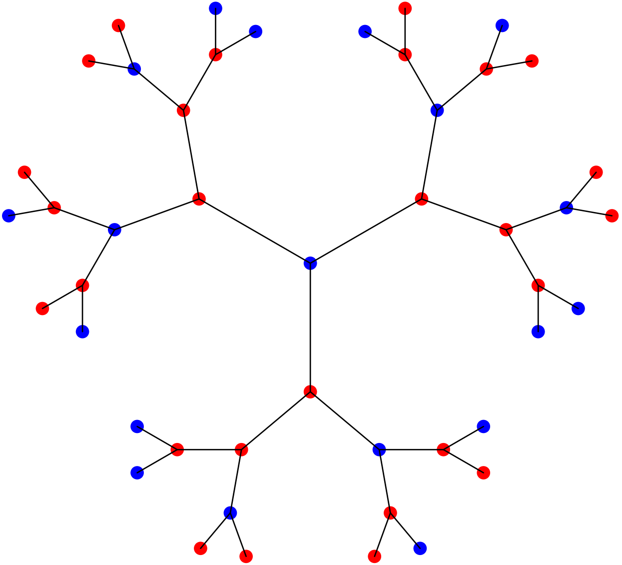
$b(2) = 4$

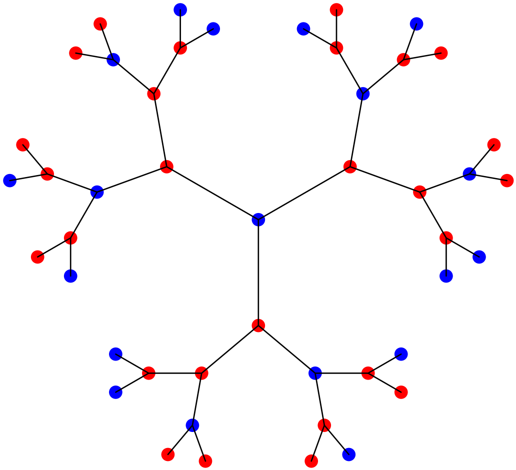


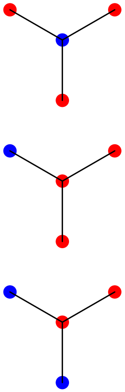
Quotient graph X:



A sturmian coloring of 3-regular tree (bounded type)







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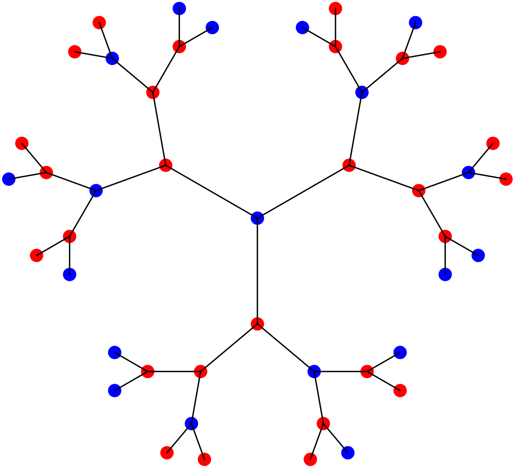

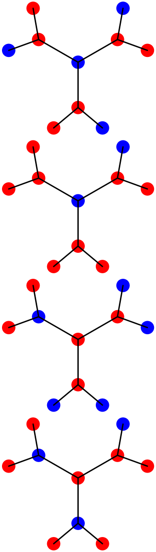

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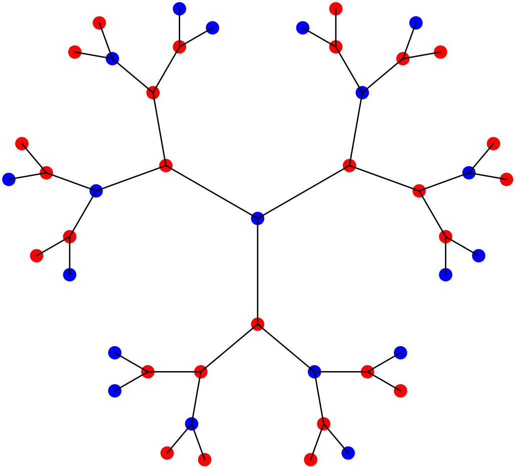

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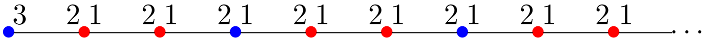
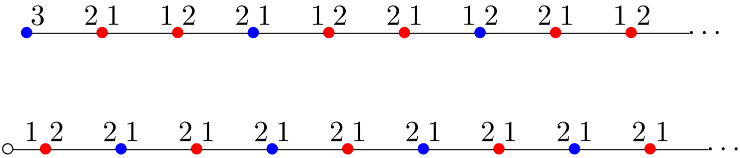
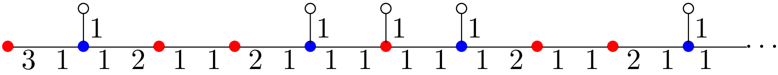


More examples of bounded type Sturmian coloring

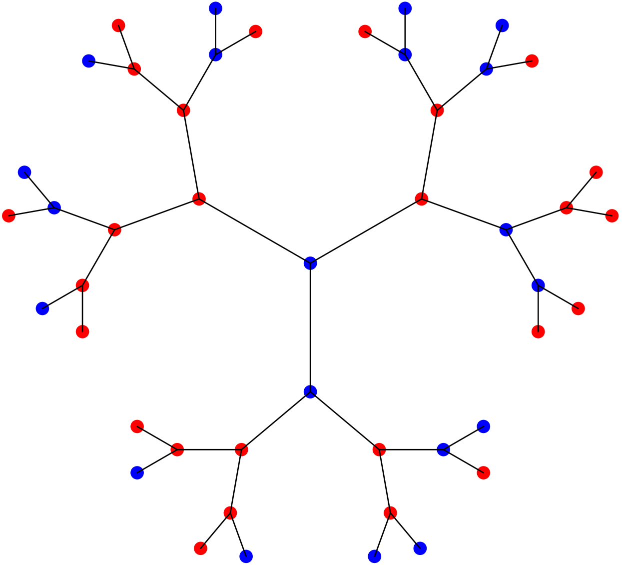
Example (Periodic configurations)

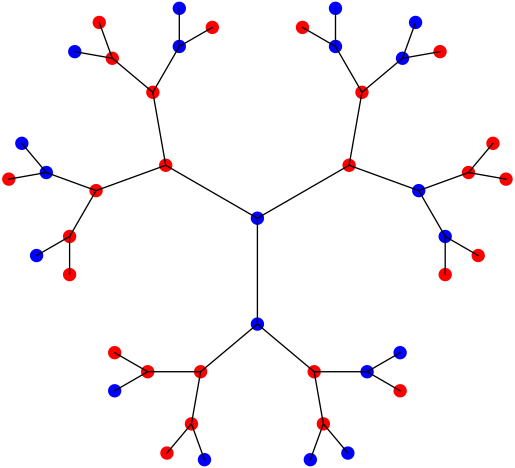


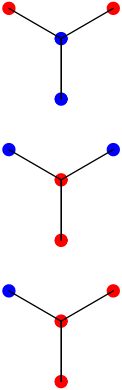
Example (Non-periodic edge configuration)



A sturmian coloring of 3-regular tree (unbounded type)







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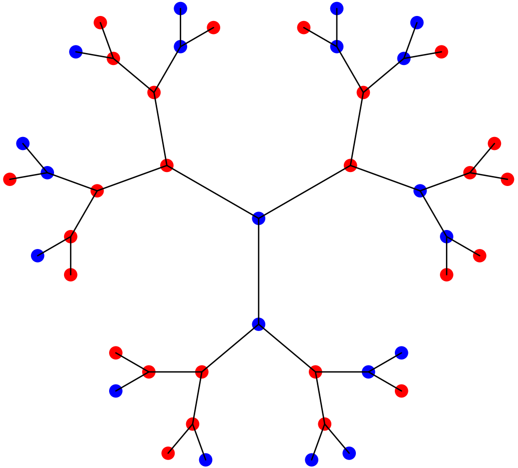

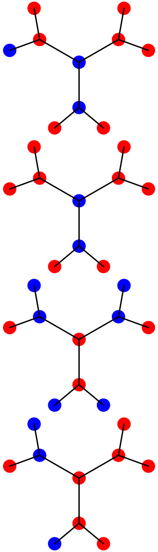

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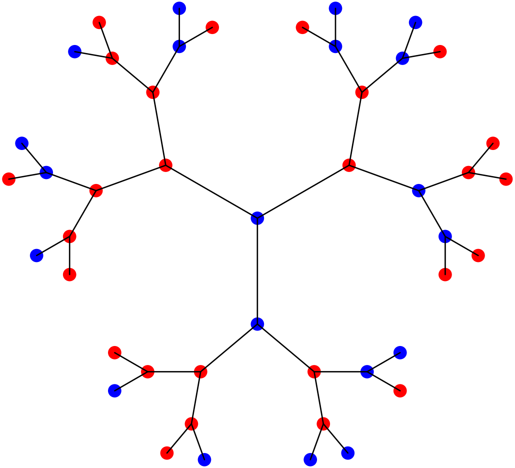
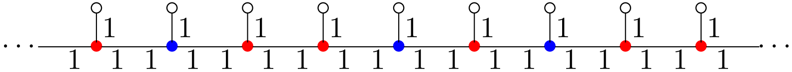
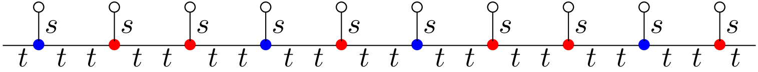


Quotient graph X:

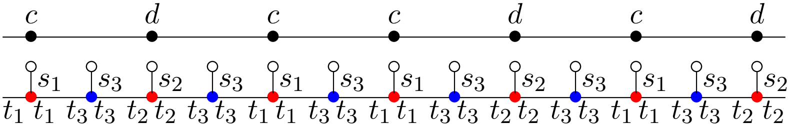


Examples of unbounded Sturmian colorings

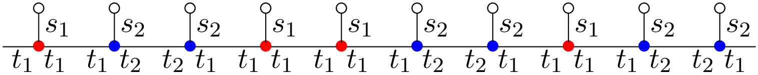
Example (with a periodic edge configuration)



Example (with a periodic vertex configuration)



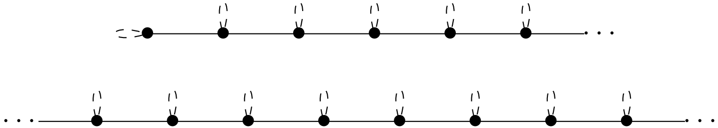
Example



Let ϕ be a Sturmian coloring of a regular tree T.

Theorem

There exists a group Γ acting on T such that ϕ is Γ-invariant, so that ϕ is a lifting of a coloring ϕ_X on the quotient graph $X = \Gamma \backslash T$. The quotient graph $X = G \backslash T$ is one of the following two types of graphs. Here, loops are expressed by dotted lines to indicate that they may exist or not.



Sturmian coloring of bounded type

A colored n-ball $[B]$ is special if there exist $x, y \in VT$ such that $[B_n(x)] = [B_n(y)] = [B]$ but $[B_{n+1}(x)] \neq [B_{n+1}(y)]$.

$$\Lambda(x) = \{n \geq 0 : [B_n(x)] \text{ is special}\}.$$

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A coloring ϕ is of bounded type if $|\Lambda(x)| < \infty$, $\forall x \in VT$.
Denote $\tau(x) = \max \Lambda(x)$.

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Theorem

If ϕ is a Sturmian coloring, then there exists a proper infinite quotient graph X of T with

$$VX = \{m, m+1, m+2, \ldots, \},$$
$$EX \subset \{[i, i+1], [i+1, i] \mid i \geq m\} \cup \{[i, i] \mid i \geq m\}$$

and a coloring ϕ_X on X such that $\phi = \phi_X \circ \pi$, where $\pi : T \to X$ is the canonical quotient map and $m = \min\{\tau(x) : x \in VT\}$.

Eventually periodic Sturmian coloring

Definition

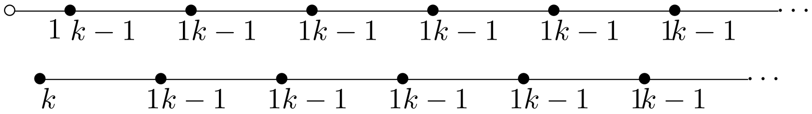
A coloring ϕ is *eventually periodic* if there exists a finite subtree K such that $T - K = \bigcup T_i$ is a finite union of subtrees T_i such that ϕ on each T_i has a periodic extension ϕ_i.

Proposition

Any eventually periodic coloring ϕ is of bounded type.

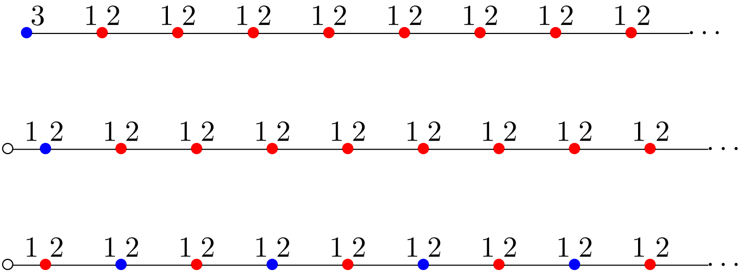
Theorem

A Sturmian coloring ϕ is eventually periodic if and only if the quotient graph X of T is one of the following graphs.



Example

The followings are eventually periodic Sturmian colorings.



Theorem

If ϕ is a Sturmian coloring of unbounded type, then there exists a proper quotient infinite graph X and a coloring ϕ_X on X such that $\phi = \phi_X \circ \pi$, where π is the projection from the regular tree T to X. Moreover, we have

$$VX = \{0, 1, 2, \ldots, \}, \quad EX \subset \{[i, i+1] \mid i \geq 0\} \cup \{[i, i] \mid i \geq 0\}$$

or

$$\begin{aligned} VX &= \{\ldots, -2, -1, 0, 1, 2, \ldots, \}, \\ EX &\subset \{[i, i+1] \mid i \in \mathbb{Z}\} \cup \{[i, i] \mid i \in \mathbb{Z}\}. \end{aligned}$$