Lazy representations, substitutions and tilings associated with complex Pisot numbers

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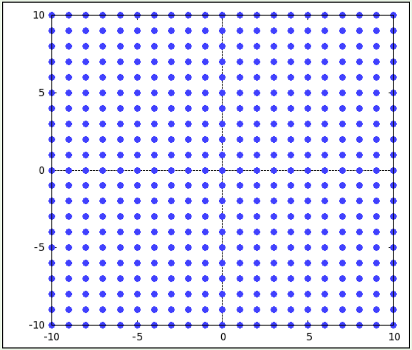
- Study of the sets $X^{\mathcal{A}}(\gamma) := \left\{ \sum_{k=0}^{N} a_k \gamma^k : a_k \in \mathcal{A}, N \in \mathbb{N} \right\} \subseteq \mathbb{Z}[\gamma]$
- The real case studied well [Erdős, Joó, Komornik, 1990]; [Akiyama, Komornik, 2011]; [Feng, to appear]; ...
- Complex case?

Couple examples

$$X^{\mathcal{A}}(\gamma) := \left\{ \sum_{k=0}^{N} a_k \gamma^k : a_k \in \mathcal{A}, N \in \mathbb{N} \right\} \subseteq \mathbb{Z}[\gamma]$$

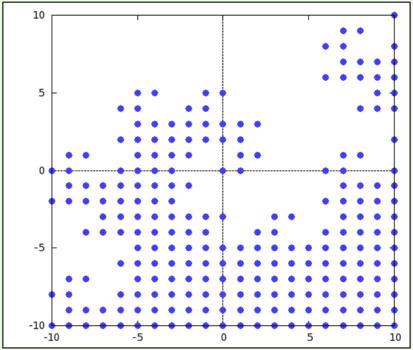
Example (Penney 1965)

Take $\mathcal{A} = \{0, 1\}$ and $\gamma = i - 1$.
Then $X^{\mathcal{A}}(\gamma) = \mathbb{Z}[i] = \mathbb{Z} + i\mathbb{Z}$.



Example

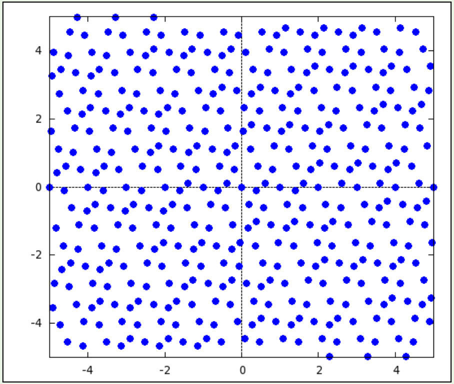
Take $\mathcal{A} = \{0, 1\}$ and $\gamma = i + 1$.
Then $X^{\mathcal{A}}(\gamma)$ is not relatively dense.



$$X^{\mathcal{A}}(\gamma) := \left\{ \sum_{k=0}^{N} a_k \gamma^k : a_k \in \mathcal{A}, N \in \mathbb{N} \right\} \subseteq \mathbb{Z}[\gamma]$$

Example (Complex Tribonacci constant)

Take $\mathcal{A} = \{-1, 0, 1\}$ and γ root of $\gamma^3 = -\gamma^2 - \gamma + 1$.



Definition (Complex Pisot number)

Algebraic integer $\gamma \in \mathbb{C} \setminus \mathbb{R}$ is a complex Pisot number if:

- $|\gamma| > 1$
- all Galois conjugates of γ but $\bar{\gamma}$ satisfy $|\gamma'| < 1$

Examples:

- Cubic case: Let $\beta \in \mathbb{R}$ be a cubic Pisot number with $\beta' \notin \mathbb{R}$. Then $1/\beta'$ is a complex Pisot number.
- All complex quadratic integers greater than one: $i - 1$, $i\sqrt{2}$, $2i$, ...

Definition (Hyperbolic complex number)

Algebraic number $\gamma \in \mathbb{C}$ is hyperbolic if

- $|\gamma| \neq 1$
- all Galois conjugates of γ satisfy $|\gamma'| \neq 1$

Theorem (Zaïmi 2005)

Consider $\gamma \in \mathbb{C}$, $|\gamma| > 1$ and $\mathcal{A} \subset \mathbb{Q}(\gamma)$, $\#\mathcal{A} \geq 2$. Then $X^{\mathcal{A}}(\gamma)$ is uniformly discrete if and only if γ is $\pm$Pisot or complex Pisot number.

Because $X^{\mathcal{A}}(\gamma) - X^{\mathcal{A}}(\gamma) = X^{\mathcal{A}-\mathcal{A}}(\gamma)$, the sets are Mayer sets for $\pm$/complex Pisot γ.

Theorem (H, Pelantova 2013; Dubickas 2011)

Suppose γ is a hyperbolic algebraic integer with $|\gamma| > 1$.

Then there exists $m \in \mathbb{N}$ such that $X^{\mathcal{A}}(\gamma)$ is relatively dense (in $\mathbb{R}$, $\mathbb{C}$), where $\mathcal{A} = \{-m, \dots, m\}$.

- Both proofs are based on polynomials with a dominant coefficient
- None of these two provides the tight bound

1. γ root of a polynomial $P(t) \in \mathbb{Z}[t]$ with a dominant coefficient $\iff$ γ hyperbolic integer [cf. Frougny et al., Akiyama et al., Burcsi]

$$P(t) = d_{n-1}t^{n-1} + \cdots + d_1 t + d_0 \quad \text{with} \quad d_j \in \mathbb{Z}$$

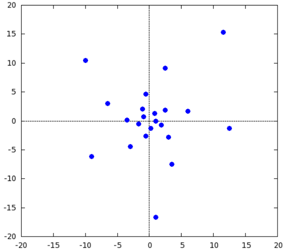
$$\exists k \quad \text{such that} \quad |d_k| > \sum_{j \neq k} |d_j|$$

2. The set $Fin_{\mathcal{A}}(\gamma)$: all numbers with finite expansions

$$a_{N-1} \cdots a_1 a_0 \bullet a_{-1} a_{-2} \cdots a_{-n}$$

3. Then addition in $Fin_{\mathcal{A}}(\gamma)$ is possible in parallel, where $\mathcal{A} = \{-d_k, \ldots, d_k\}$ [Frougny, Pelantova, Svobodova 2011]
4. Therefore $Fin_{\mathcal{A}}(\gamma)$ is closed under addition

5 We have: 0 is in interior of some triangle $z_1, z_2, z_3 \in Fin_{\mathcal{A}}(\gamma)$



6 Therefore $Fin_{\mathcal{A}}(\gamma)$ is relatively dense

7 $\frac{1}{\gamma^N} Fin_{\mathcal{A}}(\gamma) = Fin_{\mathcal{A}}(\gamma)$, therefore $Fin_{\mathcal{A}}(\gamma)$ is dense

8 Consider $z = a_{N-1} \dots a_0 \bullet \underbrace{a_{-1} \dots a_{-n}}_{\text{small}} \in Fin_{\mathcal{A}}(\gamma)$

9 Then $x = a_{N-1} \dots a_0 \bullet \in X^{\mathcal{A}}(\gamma)$ is “close to” z

10 $Fin_{\mathcal{A}}(\gamma)$ dense $\implies X^{\mathcal{A}}(\gamma)$ relatively dense

QED

QUESTION:

How to choose exactly one representation
of each point $z \in X^{\mathcal{A}}(\gamma)$?

Example (in complex Tribonacci):
$1110\bullet = 0001\bullet$

- Fix a hyperbolic algebraic integer $\gamma \in \mathbb{C}$, $|\gamma| > 1$
- Fix an alphabet $\mathcal{A} = \{-m_1, \dots, 0, \dots, m_2\} \subset \mathbb{Z}$

Definition (Lazy representation)

Let $z \in X^{\mathcal{A}}(\gamma)$. Then $a_{N-1} \cdots a_0$ is its lazy representation if

1. $a_{N-1} \neq 0$;
2. $z = a_{N-1}\gamma^{N-1} + + a_1\gamma + a_0$;
3. for all $b_{N-1} \cdots b_0$ satisfying ●, we have $|b_{N-1}| \cdots |b_1||b_0| \succeq_{lex} |a_{N-1}| \cdots |a_1||a_0|$.

Theorem (H, Steiner 2013; for complex Pisot cf. Petronio 1994)

1. *Each point in $X^{\mathcal{A}}(\gamma)$ has a unique lazy representation.*
2. *The language of the lazy representations is sofic (up to leading 0's).*

1. Zero Automaton: recognizes all finite expansions of 0 [cf. Frougny 1992]
2. This automaton is finite:
   - The states are all algebraic integers $z \in \mathbb{Z}[\gamma]$ that have finite expansions

$$a_{N-1} \cdots a_1 a_0 \bullet \qquad \text{and} \qquad \bullet\, a_{-1} a_{-2} \cdots a_{-n}$$

   - Consider the Galois conjugates of γ.
     - For $|\gamma'| > 1$ the values of all $\bullet a_{-1} a_{-2} \cdots a_{-n}$ are bounded
     - For $|\gamma'| < 1$ the values of all $a_{N-1} \cdots a_1 a_0 \bullet$ are bounded
     - (For $|\gamma'| = 1$ the values of both of them are unbounded)
   - The set

$$\left\{ (z, z', z'', \ldots, z^{(d-1)}) : z \in \mathbb{Z}[\gamma] \right\} \subset \mathbb{C}^d$$

     is a lattice
   - Intersection of a lattice with a bounded set is finite

Sketch of the proof

3. Zero Automaton $\to$ "Equality Transducer"
4. Equality Transducer $\to$ "Non-Lazy Transducer" recognizing u, v s.t.:
   - they represent the same number
   - they are not equal
   - $|u| \succeq_{lex} |v|$
5. Automata sizes: $\#\text{NLT} \leq 2\#\text{ZA}$
6. The 1st component of "Non-Lazy" relation is the non-lazy language

QED

"Definition by example":

- Base γ Golden mean
- Minimal polynomial $\gamma^2 = \gamma + 1$
- Alphabet $\mathcal{A} = \{-1, 0, 1\}$
- Zero automaton: 29 states
- Non-lazy automaton: 34 states
- Lazy automaton: 7 states

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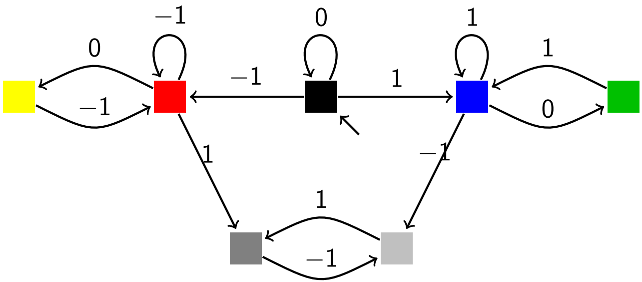

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$$x \mapsto \{\gamma x, \gamma x + 1, \gamma x - 1\}$$
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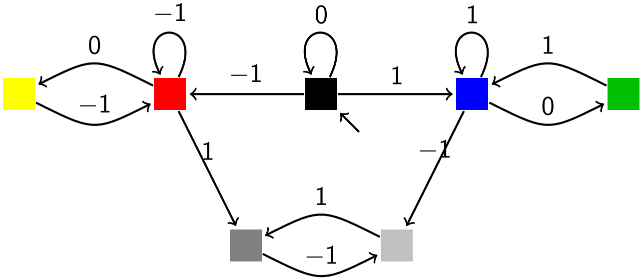
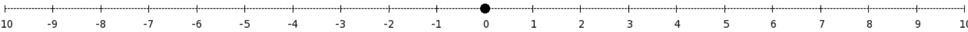


Substitutions on multi-colour sets

- Base: Golden mean $\gamma \approx 1.61803399\ldots$
- Root of $\gamma^2 = \gamma + 1$
- Real base, Pisot number
- Alphabet $\{-1, 0, 1\}$ # of colours: 7
- Number of steps of the substitution: $\mathbf{0}$



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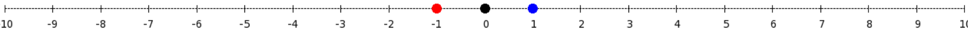
- Number of steps of the substitution: **1**



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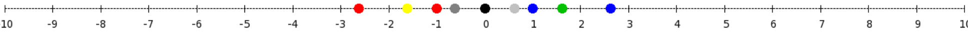
- Number of steps of the substitution: **2**



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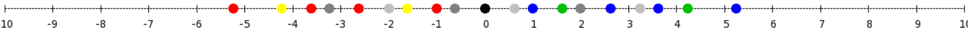

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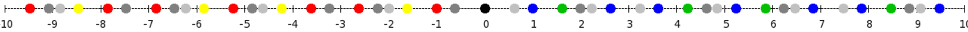

Substitutions on multi-colour sets

- Base: Complex Tribonacci constant $\gamma^3 = -\gamma^2 - \gamma + 1$
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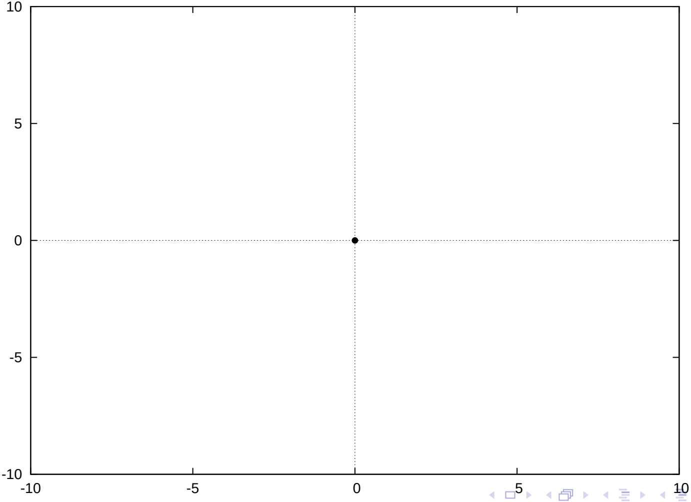

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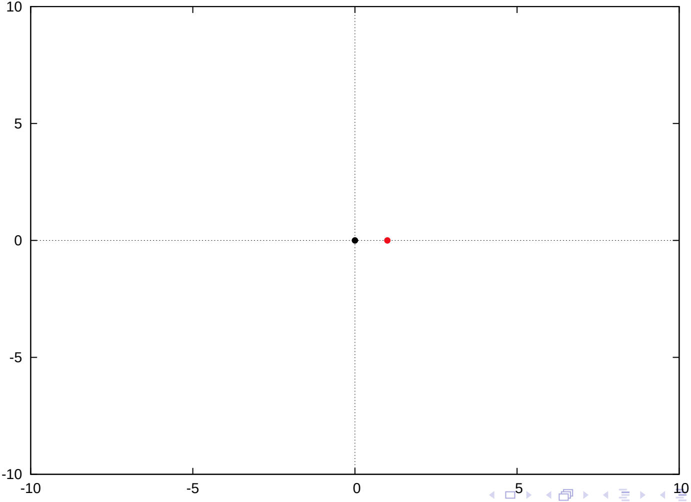

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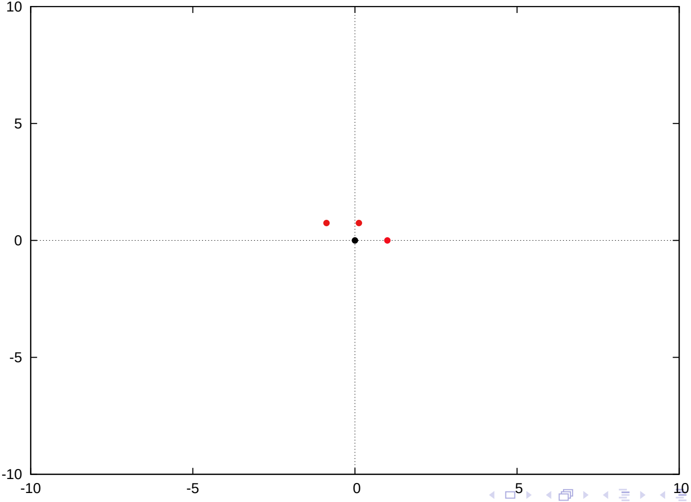

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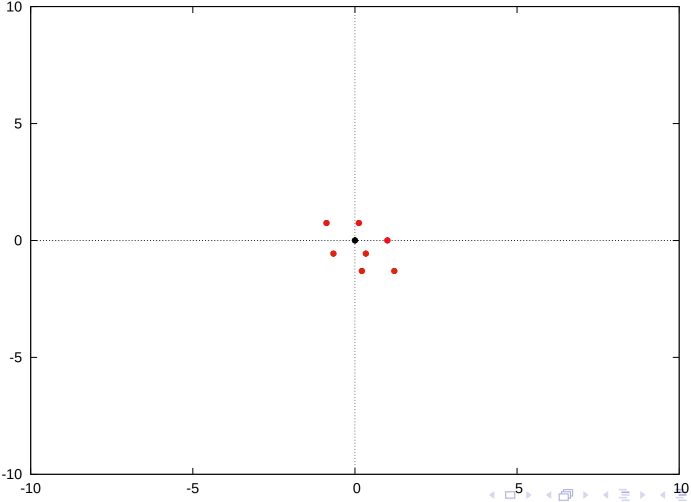

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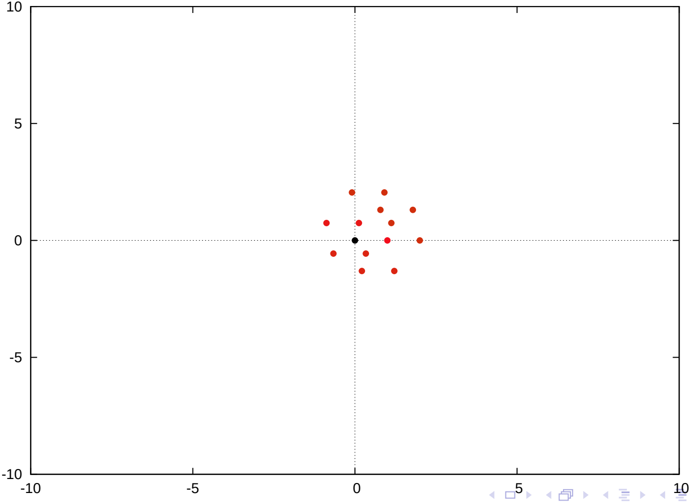
- Number of steps of the substitution: **4**



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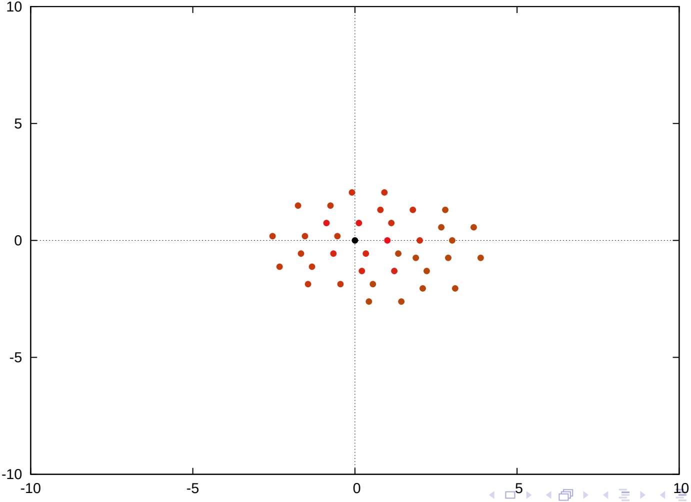
- Number of steps of the substitution: **6**



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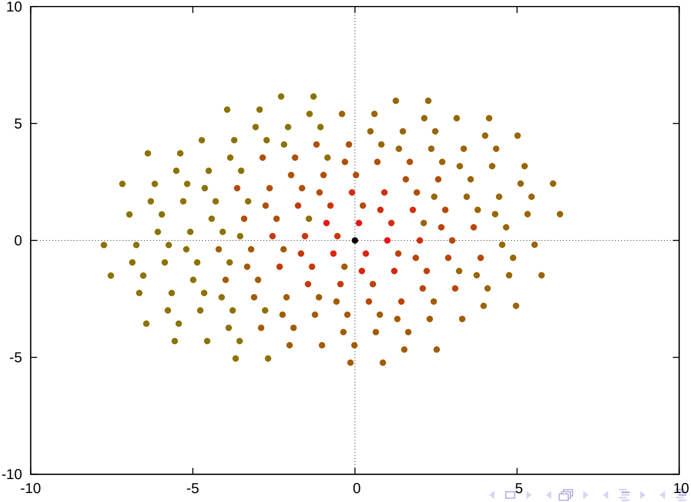

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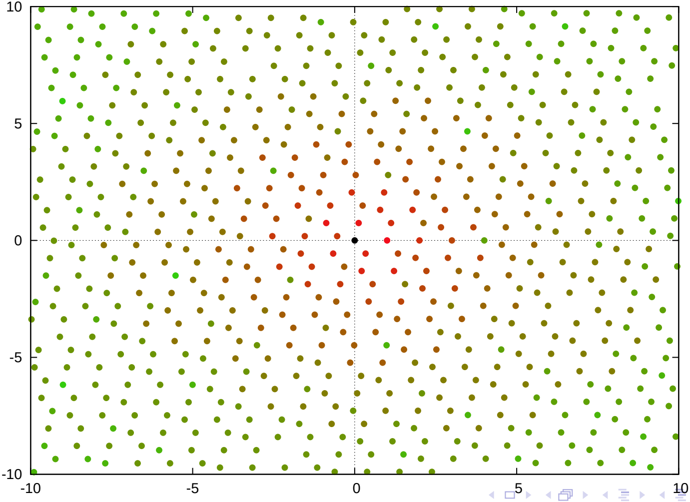
- Number of steps of the substitution: **18**



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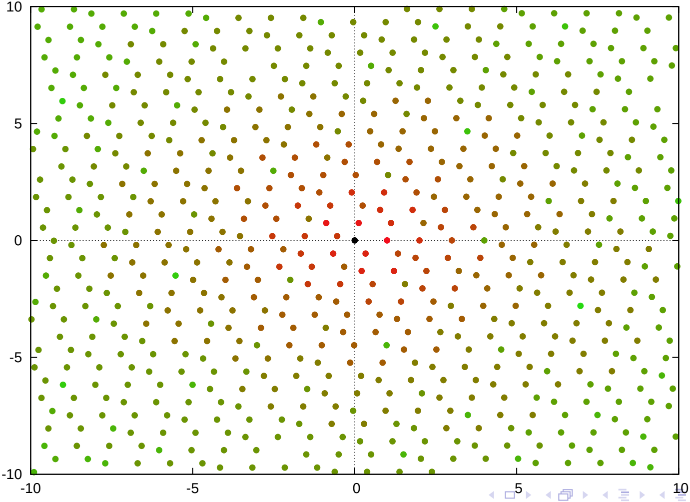
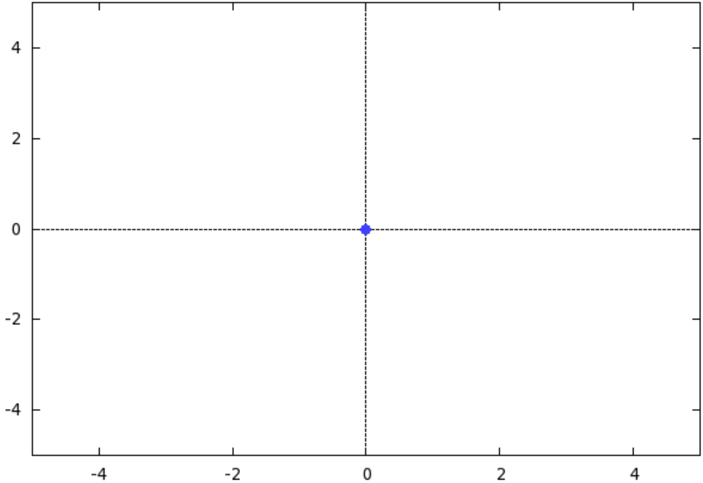


- Base: Minus 3rd root of two $\gamma = \sqrt[3]{2} \exp i\pi/3$
- Complex base, purely expanding number (not Pisot)
- Alphabet $\{-1, 0, 1\}$ $\quad$ $\#$ of colours: 27
- Number of steps of the substitution: $\mathbf{0}$



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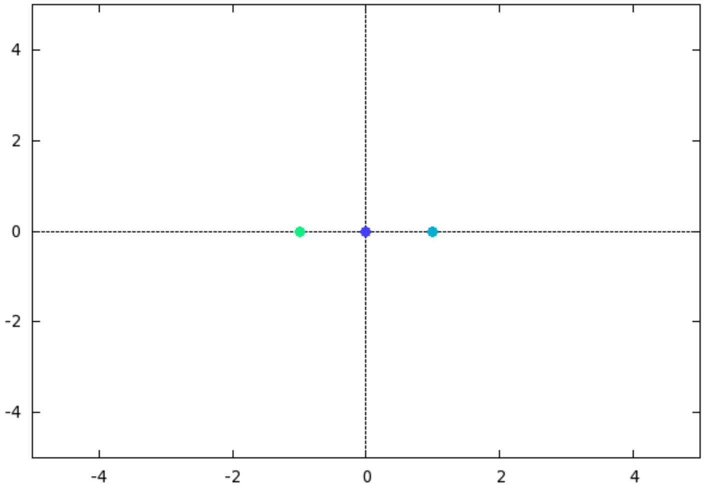

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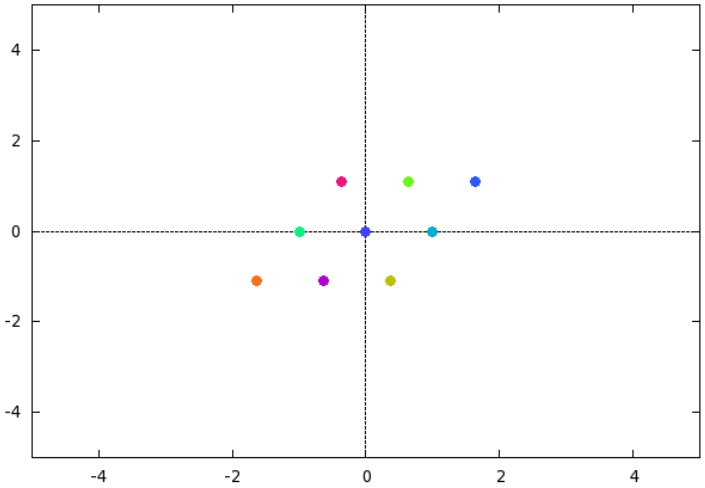

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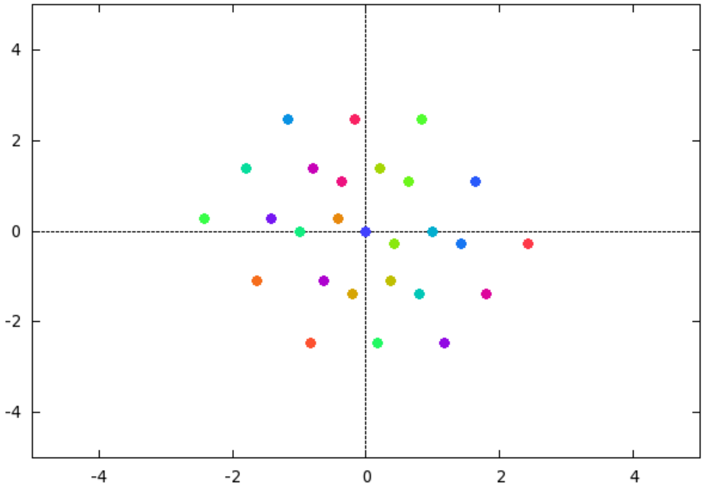

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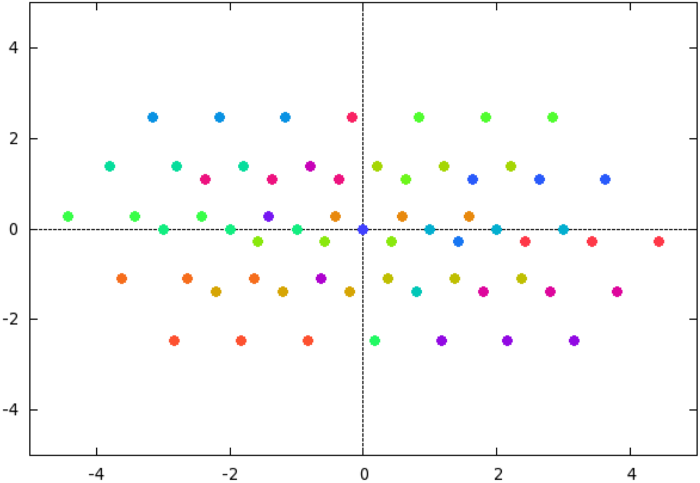

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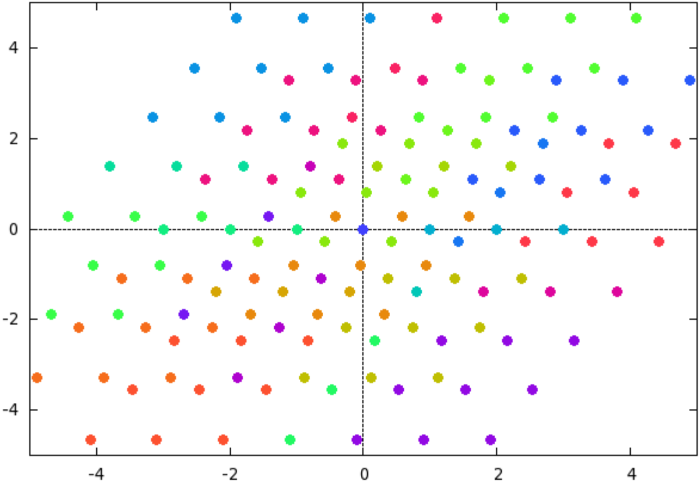

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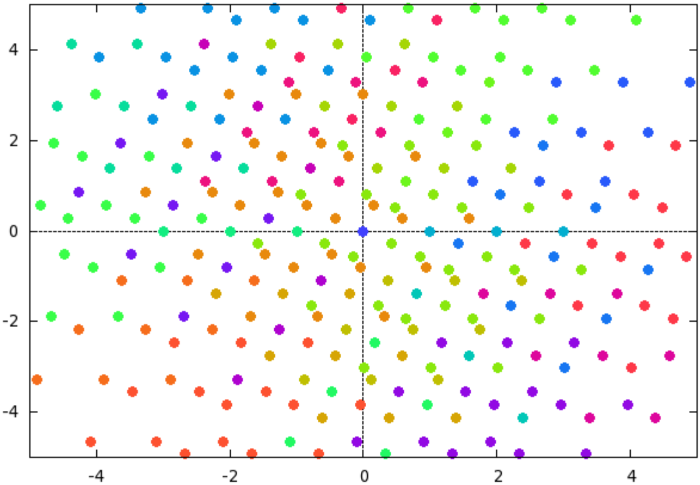

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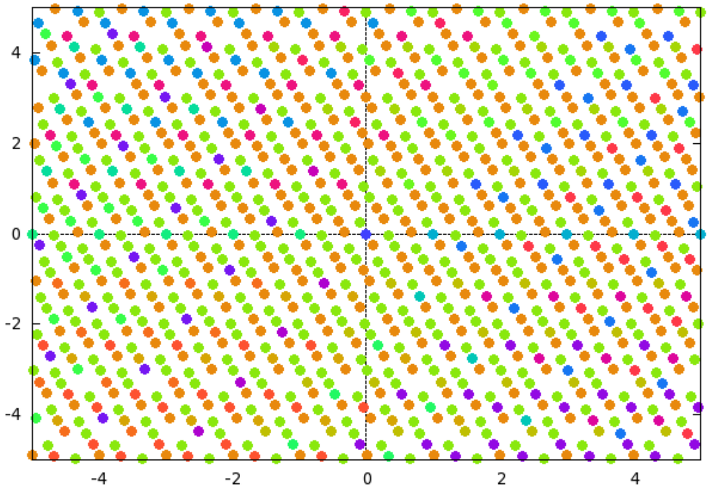

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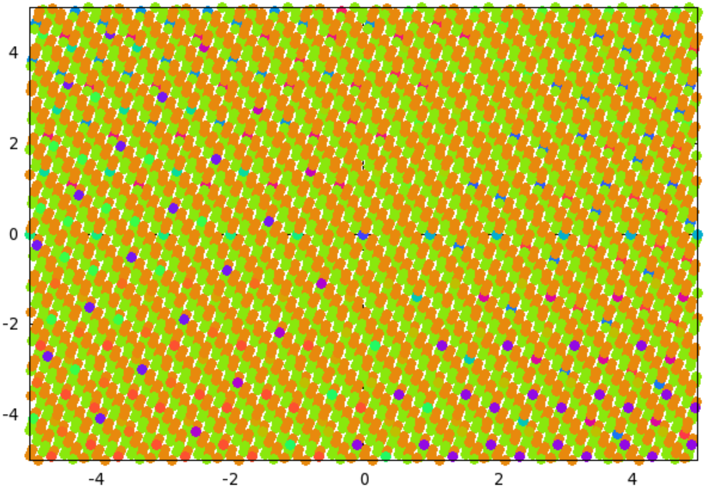
- Number of steps of the substitution: **16**

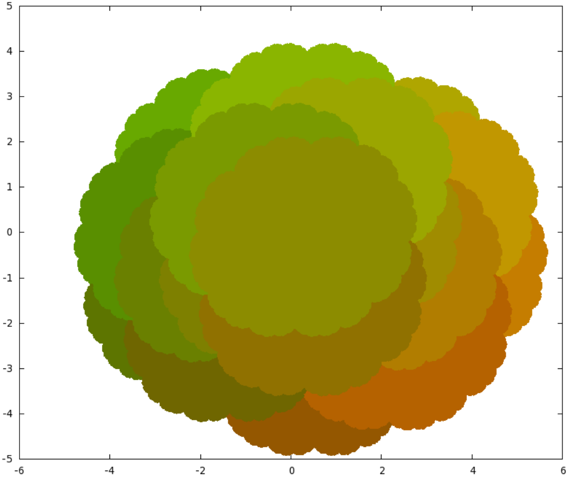


Theorem (Thurston 1989; Petronio 1994)

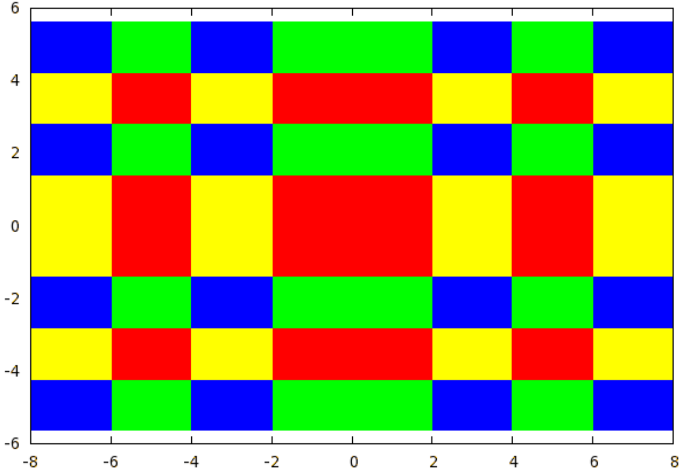
Let γ be complex Pisot number, let $\mathcal{A}$ be such that $X^{\mathcal{A}}(\gamma)$ is Delone. Then there's a tiling of $\mathbb{C}$ arising from the lazy representations.

$$T(a_{N-1}\cdots a_0\bullet) = \{a_{N-1}\cdots a_0 \bullet a_{-1}a_{-2}a_{-3}\cdots : \text{all prefixes are lazy-admissible}\}$$

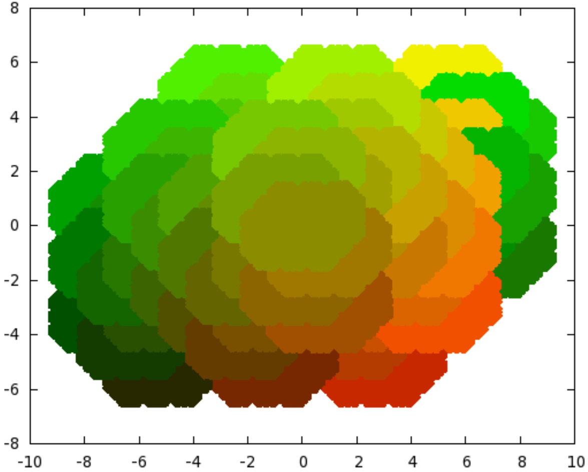
- γ complex minimal Pisot (cubic)
- Alphabet $\mathcal{A} = \{0, 1\}$



- $\gamma = i\sqrt{2}$, quadratic complex Pisot
- Alphabet $\mathcal{A} = \{-1, 0, 1\}$



- $\gamma = i - 1$, so-called Penney base
- Alphabet $\mathcal{A} = \{-1, 0, 1\}$



Open questions

1. Minimal distances in $X^{\mathcal{A}}(\gamma)$:

   How does $\{|y - z| : y, z \in X^{\mathcal{A}}(\gamma)\}$ look like?

2. Minimal alphabet:

   Minimal m such that $X^{\{-m,\dots,m\}}(\gamma)$, $X^{\{0,\dots,m\}}(\gamma)$ are Delone

3. Admissible alphabets for Delone property:

   Given complex Pisot γ and $\mathcal{A} \subseteq \mathbb{Z}, \mathbb{Z}[\gamma], \mathbb{Q}(\gamma)$, is $X^{\mathcal{A}}(\gamma)$ Delone?

4. Primitive substitutions on $X^{\mathcal{A}}(\gamma)$:

   Can we have:
   1. Sofic subshift
   2. Primitive substitution
   3. Unique representation