Matchbox Manifolds

Alex Clark *as presented by* John Hunton

University of Leicester

SubTile 2013, CIRM Marseille

Definition

A continuum $\mathcal{M}$ is a *matchbox manifold* if each point $x \in \mathcal{M}$ is contained in a neighbourhood homeomorphic to

$$\mathbb{D}^d \times \text{Totally Disconnected Space}$$

where $\mathbb{D}^d$ is the d–dimensional Euclidean disk.

Examples

- Closed manifolds
- Exceptional minimal sets of foliations
- Tiling space of a locally finite tiling
- Expanding attractor (Williams solenoids)

Historical Results

The terminology has its origins in the one–dimensional case.

Theorem

(Aarts-Martens) If a one-dimensional matchbox manifold $\mathcal{M}$ is orientable, then $\mathcal{M}$ is homeomorphic to the suspension of a homeomorphism of a totally disconnected space.

Theorem

(Fokkink) Two minimal orientable one-dimensional matchbox manifolds $\mathcal{M}_1 = \mathrm{Susp}(f_1; X_1), \mathcal{M}_2 = \mathrm{Susp}(f_2; X_2)$ are homeomorphic, if and only if there are clopen subsets $K_i \subset X_i$ such that the return maps R_i of the respective flows to K_i are conjugate.

An unorientable matchbox manifold admits an orientable double cover.

Generalizations of foliations

Definition

A *(smooth) foliated space of dimension d* is a space which admits an atlas of charts

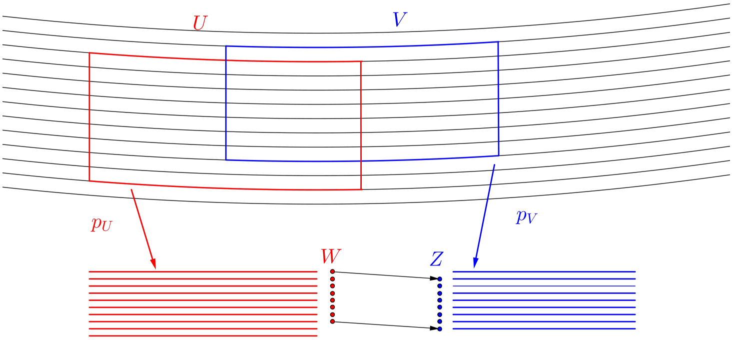
$$U \to \mathbb{R}^d \times T,$$

where each T is a subspace of the *transverse space* $\mathcal{T}$ and is such that the transition maps between charts are smooth along leaves and depend continuously on the points of transverse space.

Definition

A *smooth matchbox manifold* is a matchbox manifold admitting the structure of a smooth foliated space consistent with the matchbox structure.

The intersection of charts in a foliated space



The intersection determines a homeomorphism $W \to Z$, where W is a subspace of the transversal of U and Z is a subspace of the transversal of V.

Equicontinuity in Matchbox Manifolds

To introduce dynamics into the study of foliated spaces, we use the pseudogroup generated by the transition maps of transversals of an atlas of charts.

The starting point for our investigation of matchbox manifolds is with the tamest class after compact leaves.

Definition

A foliated space is *equicontinuous* if for any given $\varepsilon > 0$ there is a corresponding $\delta > 0$ so that any two points of a transversal within δ remain within ε under any defined application of a pseudogroup element.

One needs to show that this is an *intrinsic* definition in the sense that it is independent of the choice of charts.

Equicontinuity in Dynamics

What would one expect using results from dynamics as a model?

Theorem

(Ellis) If the group G acts equicontinuously on the compact space X, the action of G extends to a continuous action of a compact group $G \hookrightarrow K$ that acts by homeomorphisms of X.

The closure of a single G–orbit in such a setting is a minimal set of the G–action. We are assuming the image of G is dense in K here.

Corollary

A compact minimal set of an equicontinuous action is homeomorphic to a quotient $K/\mathrm{Stabilizer}$ of a compact group by a closed subgroup. In the case that G and hence K is abelian, such a minimal set supports the structure of a compact abelian group $\mathcal{G}$.

Corollary

A compact minimal set of an equicontinuous action of an $\mathbb{R}^n$ is homeomorphic to a compact, connected abelian group, and so by Pontryagin duality can be represented

$$\mathcal{G} = \varprojlim \{ G_0 \longleftarrow G_1 \longleftarrow G_2 \longleftarrow \cdots \}$$

where each G_i is a torus of the same dimension as $\mathcal{G}$ and the bonding maps are group epimorphisms.

If locally connected, $\mathcal{G}$ is a torus; otherwise $\mathcal{G}$ is a matchbox manifold.

What about the more general case?

Definition

A space X is *homogeneous* if for any two points $x, y \in X$ there is a homeomorphism $h : (X, x) \to (X, y)$.

Any space of the form $K/\mathrm{Stabilizer}$ is homogeneous.

Equicontinuous Matchbox Manifolds

Definition

An *n-dimensional (McCord) solenoid* is an inverse limit

$$\mathcal{S} = \varprojlim \{ M_0 \longleftarrow M_1 \longleftarrow M_2 \longleftarrow \cdots \}$$

where for $\ell \geq 0$, M_ℓ is a closed, n-dimensional manifold, and the bonding maps $f_{\ell+1} \colon M_{\ell+1} \to M_\ell$ are covering maps.

The basic example is of the dyadic solenoid, where each M_i is a circle and each bonding map is the doubling map of the circle.

There is a close parallel with a covering space: the projection

$$\mathcal{S} \to M_0$$

is a fiber bundle projection with fibers homeomorphic to the Cantor set.

The bundle projection $\mathcal{S} \to M_0$ of

$$\mathcal{S} = \lim_{\leftarrow} \{ M_0 \longleftarrow M_1 \longleftarrow M_2 \longleftarrow \cdots \}$$

can be thought of as being approximated by the finite coverings

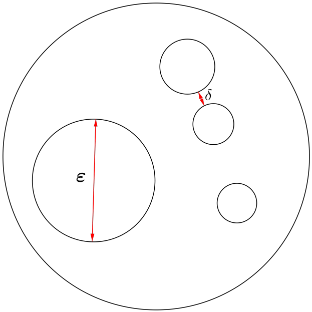
$$\overbrace{M_0 \longleftarrow M_1 \longleftarrow M_2 \longleftarrow \cdots \longleftarrow M_i}^{F_i}$$

If each F_i is a regular (normal) covering, then the bundle projection $\mathcal{S} \to M_0$ is a *principal* bundle projection with associated profinite group of bundle automorphisms:

$$\Delta = \lim_{\leftarrow} \{ \Delta_0 \longleftarrow \Delta_1 \longleftarrow \Delta_2 \longleftarrow \cdots \}$$

where Δ_i is the group of deck transformations of F_i.

The fundamental group $\pi_1(M_0)$ acts on a fiber F of $\mathcal{S} \to M_0$ in a natural way that is analogous to the monodromy action of a covering map. This action is equicontinuous.



The smaller disks represent fibers of a projection $\mathcal{S} \to M_i$ inside the large disk, which represents F. These will be clopen sets permuted by the action of $\pi_1(M_0)$ on F. Two points closer together than δ remain within ε under the action of $\pi_1(M_0)$.

The profinite group of bundle automorphisms Δ is a compactification of this $\pi_1(M_0)$ action in the case that each F_i is regular, and Δ can be naturally identified with F.

Theorem

(McCord) A solenoid $\mathcal{S}$ is homogeneous if each F_i is a regular covering map.

In the general (not regular) case, this $\pi_1(M_0)$ action extends to a compact profinite group action in natural way, only this action no longer induces a group structure on F.

The analogy with dynamics breaks down at this point:

Theorem

(Schori) There exist non-homogeneous solenoids.

Theorem

(Fokkink, Oversteegen) If a solenoid $\mathcal{S}$ is homogeneous, it can be presented as an inverse limit in which each F_i is a regular covering map.

Conjecture

(Fokkink, Oversteegen) A homogeneous matchbox manifold is either a manifold or is homeomorphic to a solenoid.

This is a natural extension of a question of Bing that was answered affirmatively by Hagopian: If every subcontinuum of a homogeneous continuum X is an arc, is X homeomorphic to a solenoid?

Theorem

(C., Hurder) A solenoid $\mathcal{S}$ is an equicontinuous matchbox manifold, where the leaves of the foliation are the path components of $\mathcal{S}$. Conversely, an equicontinuous matchbox manifold is homeomorphic to a solenoid or a manifold.

Proof sketch: If one chooses an atlas of charts that come from preimages of the projection, $\mathcal{S} \to M_0$, the first implication is similar to showing the equicontinuity of the $\pi_1(M_0)$ action on a fiber as before.

For the other implication, use the equicontinuity to define a coding that allows a decomposition of a well chosen local transverse space T into clopen sets C_i that are permuted by all possible returns to T.

Identify points in the same clopen set C_i and extend this identification to the whole space to obtain a manifold in the quotient.

Recursively refine the covering with clopen sets, obtaining at each stage a manifold quotient that covers the quotient from the previous step.

This leads to the proof of the conjecture of Fokkink and Oversteegen.

Theorem

(C., Hurder) A homogeneous smooth matchbox manifold $\mathcal{H}$ is equicontinuous, and hence is homeomorphic to a solenoid or manifold.

Proof sketch: The proof relies on an application of a theorem of Effros to the homeomorphism group of a homogeneous space:

If the compact space X is homogeneous, then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\mathrm{dist}(x, y) < \delta$ then there is homeomorphism $h : (X, x) \to (X, y)$ such that $\mathrm{dist}(h, id_X) < \varepsilon$.

Thus, for a given ε, if x and y are points in a local transversal of $\mathcal{H}$ that are sufficiently close, we can map the leaf of x to the leaf of y while not moving any point more than ε. This means that leaves which are close together cannot drift too far apart, eventually implying equicontinuity.

Abundance of Solenoids in Foliations

Theorem

(C. , Hurder) Let M_0 be a closed manifold of dimension n, with $H^1(M_0, \mathbb{R}) \neq 0$. Let $q \geq 2$, $r \geq 1$, $\epsilon > 0$, and $\mathcal{F}_0$ denote the product foliation of $M_\epsilon := M_0 \times \mathbb{B}^q_\epsilon$. Then there exists a C^r-foliation $\mathcal{F}$ of M_ϵ which is C^r-close to $\mathcal{F}_0$, such that

- *M_0 is a leaf of $\mathcal{F}$*
- *$\mathcal{F} = \mathcal{F}_0$ near the boundary of M_ϵ*
- *$\mathcal{F}$ has a minimal set $\mathcal{S}$ which is a solenoid with base M_0*
- *each leaf $L \subset \mathcal{S}$ is a covering of M_0.*

Moreover, $\mathcal{F}$ is a distal foliation.

By Langevin and Rosenberg's extension of Thurston-Reeb Stability, when $H^1(M_0, \mathbb{R}) = 0$ some version of the product structure persists under small perturbations.

In all the examples we construct, the bundle automorphism group Δ of the solenoid $\mathcal{S}$ is abelian.

General Matchbox Manifolds

For a general matchbox manifold, the pseudogroup of transitions maps will not permute the clopen sets in a transversal.

In a tiling space, a clopen set in the transversal represents a certain configuration of tiles around the origin and a permutation of the clopen sets in the transversal would correspond to periodicity of the tiling.

In general, the clopen sets will intermingle in complicated ways under the action of the pseudogroup, which leads to branching in the approximates and expansive transverse dynamics.

To obtain nice inverse limit expansion of matchbox manifolds, one can use a generalization of the coding technique indicated above for the equicontinuous case, as begun in the work of Gambaudo and Martens.

Theorem

(C., Lukina, Hurder) A minimal smooth matchbox manifold $\mathcal{M}$ can be presented as an inverse limit

$$\mathcal{M} = \varprojlim \{ M_0 \longleftarrow M_1 \longleftarrow M_2 \longleftarrow \cdots \}$$

with triangulated branched manifolds M_i and simplicial bonding maps.

The bonding maps will additionally preserve the dimension of the cells in the associated simplicial complexes.

This complements the results of Cuesta, Rojo and Stadler; and Rojo, where similar presentations were found using zooming techniques.

These presentations open the door of shape theory to the study of matchbox manifolds that in much the same way that the presentation of Anderson-Putnam, Gähler, Sadun, Barge-Diamond-Hunton-Sadun did for the study of tiling spaces.

Shape Theory

Loosely: in shape theory one identifies topological spaces with their expansions as inverse systems of CW complexes (but not their limits).

Many of the important invariants of tiling spaces are also shape invariants: Čech cohomology and various forms of K-theory.

With Hunton, we developed a shape invariant the L–invariant.

While we used this invariant in the context of tiling spaces, the same concepts carry over to general matchbox manifolds.

The primary application we have found for this invariant is in the following theorem.

Theorem

(C.,Hunton) If a tiling space Ω_P admits a codimension one embedding in a manifold, then the L–invariant for Ω_P vanishes.

This should admit a generalization to all matchbox manifolds.

Attractors and Tiling Spaces

Our study led to a link between two of the better studied classes of matchbox manifolds.

Theorem

(C.,Hunton) Let A be a codimension 1 expanding attractor of the diffeomorphism $h\colon M \to M$. If A is orientable, then it is shape equivalent to a $(d+1)$-dimensional torus with a finite number of points removed, $\mathbb{T}^{d+1} - \{k\}$ say. If A is unorientable, it is shape equivalent to a polyhedron that has a 2 to 1 cover by some $\mathbb{T}^{d+1} - \{k\}$.

Theorem

(C.,Hunton) Every oriented codimension 1 expanding attractor A in the $(d+1)$-dimensional manifold M is homeomorphic to the tiling space Ω_P of an aperiodic tiling P of $\mathbb{R}^d$. In the case $d = 1$ we may choose P to be given by a primitive substitution; for $d \geqslant 2$, we can describe P as a projection tiling.

One of the challenges facing us in the study of general matchbox manifolds is how to compute the shape invariants effectively.

Now that techniques are being developed to find inverse limit presentations of matchbox manifolds, we need to find ways of relating the dynamics and cohomology as we have seen done with tiling spaces.

The next talk shall address how we can use equicontinuous matchbox manifolds to better understand the general matchbox manifolds.