Spectral Analysis of $\mathbb{Z}^d$ Substitution Tilings

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    - Rudin-Shapiro, and higher dimensional generalizations due to Frank give examples with Lebesgue components in the spectrum.
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  - Rudin-Shapiro, and higher dimensional generalizations due to Frank give examples with Lebesgue components in the spectrum.
  - Recent results of Baake and Grimm prove that the spectrum of all bijective $\mathbb{Z}^d$-substitutions on 2 symbols are singular to Lebesgue.
  - Queffelec gave an example of a bijective constant length substitution which she claims to have Lebesgue component.

In her book *Substitution Dynamical Systems - Spectral Analysis*, Queffelec develops a number of techniques for studying the spectrum of constant length substitutions (our $d = 1$ case). We have generalized many of these methods for application to $\mathbb{Z}^d$ substitutions (of *constant length*). In this talk, we will:

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1. describe two of her constructions which are very useful for spectral analysis: the *coincidence matrix*, and the *ergodic classes* of a substitution,
2. describe an algorithm, based on these constructions, which is effective for determining singularity to Lebesgue measure for the spectrum of a primitive $\mathbb{Z}^d$-substitution.

Notation and Convention

- Write $\mathcal{A} = \{1, \ldots, s\}$, our *alphabet*. Assume $s \geq 2$.

  We will use greek letters α, β, γ, δ to denote letters in $\mathcal{A}$.

- Take $d \geq 1$, and fix $\mathbf{q} > \mathbf{1}$ in $\mathbb{Z}^d$, the *expansion* factor.

  Operations and relations in $\mathbb{Z}^d$ will be interpreted *component wise*, elements in **bold font**.

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- Let $\mathcal{A}^{\mathbb{Z}^d}$ denote the collection of all $\mathbb{Z}^d$-sequences with coefficients in $\mathcal{A}$.
- Let $T : \mathbf{a} \mapsto T_{\mathbf{a}}$ denote the $\mathbb{Z}^d$-action of translation for sequences in $\mathcal{A}^{\mathbb{Z}^d}$.
- Then $(\mathcal{A}^{\mathbb{Z}^d}, T)$ denotes the full $\mathbb{Z}^d$-shift on $\mathcal{A}$, with the usual topology.

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- For $\mathbf{j} \in \mathbb{Z}^d$ define $\mathcal{I}(\mathbf{j}) := \left\{ \mathbf{k} \in \mathbb{Z}^d : \mathbf{0} \leq \mathbf{k} < \mathbf{j} \right\}$
  - Alternatively, $\mathcal{I}(\mathbf{j}) := [0, j_1) \times [0, j_2) \times \cdots \times [0, j_d) \cap \mathbb{Z}^d$.

Definition (Substitution)

A **q**-Substitution $\mathcal{S}$ is a map

$$\mathcal{A} \to \mathcal{A}^{\mathcal{I}(\mathbf{q})} \quad \text{extended to a map} \quad \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$$

by the familiar *inflate and subdivide* rule using the map $\mathbf{t} \mapsto \mathbf{q}\mathbf{t}$.

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The composition matrix of a substitution $\mathcal{S}$, denoted $M_{\mathcal{S}}$, is the $s \times s$ matrix whose $\alpha\beta$ entry

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A substitution is *primitive* provided its composition matrix is primitive.

Let's consider two examples:

Example (Queffelec's ζ)

This is a 3-Substitution with $s = 3$, and

$$1 \mapsto 1 \quad 1 \quad 2 \qquad\qquad 2 \mapsto 2 \quad 3 \quad 3 \qquad\qquad 3 \mapsto 3 \quad 2 \quad 1$$

$$\text{with composition matrix: } M_\zeta = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$$

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Example (The Domino, or Table Substitution)

This is a plane tiling, encoded by Robinson as a $(2, 2)$-Substitution with $s = 4$:

$$1 \mapsto \begin{matrix} 4 & 1 \\ 2 & 1 \end{matrix} \qquad 2 \mapsto \begin{matrix} 2 & 2 \\ 1 & 3 \end{matrix} \qquad 3 \mapsto \begin{matrix} 3 & 4 \\ 3 & 2 \end{matrix} \qquad 4 \mapsto \begin{matrix} 1 & 3 \\ 4 & 4 \end{matrix}$$

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Observe that both examples are bijective and primitive.

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Let $X_{\mathcal{S}}$ be the collection of $\mathbb{Z}^d$-sequences whose language is contained in the set of supertiles for the substitution, that is:

$$X_{\mathcal{S}} := \{\mathcal{U} \in \mathcal{A}^{\mathbb{Z}^d} : \forall B \subset \mathbb{Z}^d \text{ finite}, \exists \gamma \in \mathcal{A}, \mathbf{j} \in \mathbb{Z}^d, n > 0 \text{ with } \mathcal{U}\big|_B = \mathcal{S}^n(\gamma)_{\mathbf{j}+B}\}$$

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Then $(X_{\mathcal{S}}, T)$ is a topological dynamical system, the *subshift* for $\mathcal{S}$.

- This coincides with taking the orbit-closure of a substitution sequence inside the full-shift.
- When $\mathcal{S}$ is primitive, we know that the subshift is uniquely ergodic. We assume this, and denote the unique invariant measure by μ.
- The measure of the initial cylinders in $X_{\mathcal{S}}$ is given by the Perron-Frobenius eigenvector for $M_{\mathcal{S}}$ corresponding to $Q := q_1 q_2 \cdots q_d$.

Spectral Data

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Let $\sigma_{\alpha\beta} \in \mathcal{M}(\mathbb{T}^d)$ denote the spectral measures for the pairs $\mathbb{1}_{[\alpha]}, \mathbb{1}_{[\beta]}$:

$$\widehat{\sigma}_{\alpha\beta}(\mathbf{k}) := \int_{X_{\mathcal{S}}} \mathbb{1}_{[\alpha]} \circ T^{\mathbf{k}} \cdot \overline{\mathbb{1}_{[\beta]}} d\mu, \quad \text{for } \mathbf{k} \in \mathbb{Z}^d.$$

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Using unique ergodicity of $(X_{\mathcal{S}}, T, \mu)$, we can also compute the $\widehat{\sigma}_{\alpha\beta}(\mathbf{k})$ as:

$$\lim_{n\to\infty} \frac{1}{Q^n} \mathrm{Card}\{\mathbf{j} + \mathbf{k} \in \mathcal{I}(\mathbf{q}^n) : \mathcal{S}^n(\gamma)_{\mathbf{j}+\mathbf{k}} = \alpha, \text{ and } \mathcal{S}^n(\gamma)_{\mathbf{j}} = \beta\}.$$

Using Riesz products, Queffelec uses this identity to establish recursion relations on the Fourier coefficients of the correlation measures:

$$\widehat{\sigma}_{\alpha\beta}(\mathbf{aq}+\mathbf{b}) \text{ in terms of } \widehat{\sigma}_{\gamma\delta}(\mathbf{a}+\mathbf{k}) \text{ for } \gamma,\delta\in\mathcal{A},\mathbf{k}\in\mathcal{I}(\mathbf{2}).$$

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Example (The 3-substitution, ζ)

Writing the $\widehat{\sigma}_{\alpha\beta}$ in matrix form as $\widehat{\Sigma}$, we have:

$$\widehat{\Sigma}(0) = \frac{1}{3}\begin{pmatrix}1 & & \\ & 1 & \\ & & 1\end{pmatrix},\ \widehat{\Sigma}(1) = \frac{1}{39}\begin{pmatrix}5 & 6 & 2\\ 6 & 2 & 5\\ 2 & 5 & 6\end{pmatrix},\ \widehat{\Sigma}(2) = \frac{1}{117}\begin{pmatrix}7 & 7 & 25\\ 25 & 7 & 7\\ 7 & 25 & 7\end{pmatrix}$$

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Theorem (Queffelec)

If $\mathcal{S}$ is a primitive and aperiodic $\mathbf{q}$-substitution on $\mathcal{A}$, then

$$\sigma_{max} \sim \omega_{\mathbf{q}} * \sum_{\alpha \in \mathcal{A}} \sigma_{\alpha\alpha}.$$

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Due to this fact, we can use the Fourier recursions for the $\sigma_{\alpha\beta}$ to determine properties of $\sigma_{\max}$. Using some constructions of Queffelec, however, we can do considerably better than this.

Given a **q**-substitution $\mathcal{S}$ on $\mathcal{A}$, let $\mathcal{A}^2$ denote the alphabet of letter pairs in $\mathcal{A}$, denoting elements by $\alpha\beta$ or $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ as benefits the situation.

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Definition

The *bisubstitution* for $\mathcal{S}$ is the $\mathbf{q}$-Substitution on $\mathcal{A}^2$ that takes the letter $\gamma\delta \in \mathcal{A}^2$ to the $\mathcal{I}(\mathbf{q})$-array of letters whose $\mathbf{k}$-th element is $\mathcal{S}(\gamma)_{\mathbf{k}}\mathcal{S}(\delta)_{\mathbf{k}}$.

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Example (Queffelec's 3-Substitution ζ)

- The alphabet $\mathcal{A}^2$ consists of: $\binom{1}{1}$ $\binom{1}{2}$ $\binom{1}{3}$ $\binom{2}{1}$ $\binom{2}{2}$ $\binom{2}{3}$ $\binom{3}{1}$ $\binom{3}{2}$ $\binom{3}{3}$

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- Substitution rule:

$$\binom{1}{1} \mapsto \binom{1}{1}\binom{1}{1}\binom{2}{2} \qquad \binom{1}{2} \mapsto \binom{1}{2}\binom{1}{3}\binom{2}{3} \qquad \binom{1}{3} \mapsto \binom{1}{3}\binom{1}{2}\binom{2}{1}$$

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Now, for each $\gamma\delta \in \mathcal{A}^2$, let $E^{\gamma\delta}$ be the collection of all letter pairs $\alpha\beta \in \mathcal{A}^2$ which appear in bisubstituted supertiles on $\gamma\delta$. In other words,

$$E^{\gamma\delta} := \left\{ \alpha\beta \in \mathcal{A}^2 : \exists N > 0, \mathbf{k} \in \mathcal{I}(\mathbf{q}^N) \text{ with } \mathcal{S}^N(\gamma)_\mathbf{k} = \alpha \text{ and } \mathcal{S}^N(\delta)_\mathbf{k} = \beta \right\}$$

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Definition

Let $E_1, E_2, \ldots, E_K$ denote the minimal elements of $\{E^{\gamma\delta} : \gamma\delta \in \mathcal{A}^2\}$, when partially ordered under set inclusion. They are the *ergodic classes* of $\mathcal{S}$, and $T := \mathcal{A}^2 \setminus \cup_1^K E_j$ is the *transient part*.

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- In the case of trivial height, $K = 1$ if and only if $\sigma_{\max}$ is pure discrete.

Example (The 3-Substitution, ζ)

We have:

$$\begin{aligned}
\zeta^2(1) &= 1 \quad 1 \quad 2 \quad 1 \quad 1 \quad 2 \quad 2 \quad 3 \quad 3\\
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From this, we can compute the $E^{\alpha\beta}$, and:

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We note here two properties which are true in more generality: E^{11} is always an ergodic class, and if the substitution is bijective, the transient part is empty (the ergodic classes partition $\mathcal{A}^2$).

Definition

The composition matrix for the bisubstitution of $\mathcal{S}$, denoted $C := C_{\mathcal{S}}$, is called the *coincidence matrix* of $\mathcal{S}$. In $\mathcal{A}^2$-coordinates,

$$C_{\alpha\beta,\gamma\delta} := \mathrm{Card}\{\mathbf{k} \in \mathcal{I}(\mathbf{q}) : \mathcal{S}_{\mathbf{k}}(\gamma) = \alpha, \text{ and } \mathcal{S}_{\mathbf{k}}(\delta) = \beta\}.$$

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- With the basis ordered this way, C has the following form:

$$\begin{pmatrix} C_1 & & & & & N_1 \\ & C_2 & & & & N_2 \\ & & \cdot & & & \cdot \\ & & & \cdot & & \cdot \\ & & & & C_K & N_K \\ & & & & & C_T \end{pmatrix}$$

Let $F := \{\nu \in \mathbb{C}^{s^2} : \nu C = Q\nu\}$, the left Q-eigenspace of C.

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Writing $\nu \gg 0$ if $(\nu_{\alpha\beta})_{\alpha,\beta\in\mathcal{A}}$ is positive definite in $M_s(\mathbb{C})$, we have

Proposition

For the primitive **q**-substitution $\mathcal{S}$, let

$$\mathcal{K} := \{\nu \in F : \nu \gg 0 \text{ and } \nu_1 = 1\}.$$

Then $\mathcal{K} \subset \mathbb{C}^{s^2}$ is convex with K extreme points, denoted

$$\text{ext}(\mathcal{K}) = \{\nu^0, \nu^1, \ldots, \nu^{K-1}\}.$$

Theorem (Queffelec)

For the primitive and aperiodic $\mathbf{q}$*-substitution* $\mathcal{S}$ *on* $\mathcal{A}$*, let* $\nu^1, \ldots, \nu^K$ *be the extreme points of* $\mathcal{K}$ *corresponding to the ergodic class decomposition for* $\mathcal{S}$*.*

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$$\lambda_j := \sum_{i=1}^{K} (\nu^j)_i \sum_{\alpha\beta \in E_i} \sigma_{\alpha\beta} = \sum_{\alpha\beta} (\nu^j)_{\alpha\beta} \sigma_{\alpha\beta}.$$

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- Using the recursive relations for the $\sigma_{\alpha\beta}$, we can compute the Fourier coefficients of each λ_j.
- Lebesgue measure m_d is ergodic for the $\mathbf{q}$-shift; thus we can conclude that $\lambda_j \perp m_d$ by checking if $\widehat{\lambda_j}(\mathbf{k}) \neq 0$ for some $\mathbf{k} \neq \mathbf{0}$.

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Results and Directions

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- All bijective primitive constant length substitutions for $d = 1$, $s = 2, 3, 4$ with $q = 2, 3, 4$.

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As ζ has two ergodic classes with no transient part, we can write $\nu \in \mathcal{K}$ as

$$\nu = \begin{pmatrix} \nu_{11} & \nu_{12} & \nu_{13} \\ \nu_{21} & \nu_{22} & \nu_{23} \\ \nu_{31} & \nu_{32} & \nu_{33} \end{pmatrix} = \begin{pmatrix} 1 & \nu_2 & \nu_2 \\ \nu_2 & 1 & \nu_2 \\ \nu_2 & \nu_2 & 1 \end{pmatrix}$$

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Thus

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Clearly, $m \perp \lambda_1$. However, using the Fourier coefficients computed earlier,

$$\widehat{\Sigma}(2) = \frac{1}{117} \begin{pmatrix} 7 & 7 & 25 \\ 25 & 7 & 7 \\ 7 & 25 & 7 \end{pmatrix}, \text{ we have that } \widehat{\lambda_2}(2) = 1 - \frac{3}{2} \cdot \frac{96}{117} \neq 0.$$

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Thus, $\lambda_2 \perp m$, so that $\sigma_{\max}$ for ζ is singular to Lebesgue measure.

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Summary of the Algorithm

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   - This can essentially be done by forcing the eigenvalues of $(\nu_{\alpha\beta})_{\alpha,\beta\in\mathcal{A}}$ with the entries ν_j in place of $\nu_{\alpha\beta}$ to be positive, noting that the transient pairs can be computed from the ν_j.

Summary of the Algorithm

Given a primitive and aperiodic $\mathbf{q}$-substitution $\mathcal{S}$ on an alphabet $\mathcal{A}$:

1. Compute the composition matrix, $M_{\mathcal{S}}$; use this to compute the $\widehat{\sigma}_{\alpha\beta}(\mathbf{0})$.
2. Using Fourier Recursion, compute $\widehat{\sigma}_{\alpha\beta}(\mathbf{k})$ for some $\mathbf{k} \neq \mathbf{0}$.
   - It is possible that computing for $\mathbf{k} \in \mathcal{I}(\mathbf{q}+\mathbf{1})$ suffices.
3. Compute the ergodic classes $\{E_1, \ldots, E_K\}$.
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5. Check whether $\widehat{\lambda}_j(\mathbf{k}) \neq 0$, if so: $\lambda_j \perp m_d$.

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5. Check whether $\widehat{\lambda}_j(\mathbf{k}) \neq 0$, if so: $\lambda_j \perp m_d$.
6. If $\lambda_j \perp m_d$ for all j, we would have $\sigma_{\max} \perp m_d$.

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Thank you for listening!